A STUDY OF SOME SPECIAL TYPE I-CONVERGENT SEQUENCE SPACES

THESIS

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RAMI KAMEL AHMAD RABABAH

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DR. VAKEEL AHMAD KHAN

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002 (INDIA)

2016

Dr. Vakeel Ahmad Khan
(Senior Assistant Professor)

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202 002 (U.P.), INDIA
e-mail: vakhanmaths@gmail.com

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Certificate

This is to certify that the contents of this thesis entitled **"A Study of Some Special Type I-convergent Sequence Spaces"** *is an original research work of* **Mr. Rami Kamel Ahmad Rababah** *under my supervision and guidance.*

I further certify that the work of this thesis, either partially or fully, has not been submitted to any other university or institution for the award of any other degree.

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DEPARTMENT OF MATHEMATICS
A.M.U., ALIGARH

Dr. Vakeel Ahmad Khan
(Supervisor)

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Rami Kamel Ahmad Rababah

Rami

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PREFACE

Convergence of sequences has always remained a subject of interest to the researchers. Several new types of convergence of sequences were introduced and studied by the researchers and named it as usual convergence, uniform convergence, strong convergence, week convergence etc. Later on, the idea of statistical convergence came into existence which is the generalization of usual convergence. Statistical convergence has several applications in different fields of Mathematics like Number Theory, Trigonometric Series, Summability Theory, Probability Theory, Measure Theory, Optimization and Approximation Theory. The notion of Ideal convergence(I-convergence) is a generalization of the statistical convergence and equally considered by the researchers for their research purposes since its inception.

In this thesis, our aim is to introduce certain I-convergent sequence spaces and study their some algebraic and topological properties, inclusion relations, decomposition theorems and some other results on these spaces.

The structure of this text is straightforward. There are six chapters devoted to the various aspects of the theory. Each chapter is divided into sections. The numbers in the square brackets refers to the references listed in the bibliography. Equations are indicated in small brackets whereas definitions, examples, remarks and results are enlisted separately and freely with consecutive numbers of the respective chapter.

As usual **chapter one** contains notations and conventions, basic definitions, examples and some important well known results related to our study which are required for the development of the subject in the subsequent chapters. This chapter is an attempt to make this thesis as self contained as possible. Some known sequence spaces like ω, c_0, c, ℓ_∞, $c_0(p)$, $c(p)$, $\ell_\infty(p)$ etc. where $p = (p_K)$ is sequence of positive real numbers unless and otherwise stated, that motivated us for our work to generalize them in various directions via I-convergence are also given in this chapter. The main concepts that defined in this chapter and remained instrumental to us in the subsequent chapters are normed spaces, Banach spaces, paranormed spaces, modulus function, Orlicz function, Lipschitz function, monotonicity, solidity, symmetricity etc. of sequence spaces, quasilinear space, statistical convergence of sequences etc. This chapter concludes with an introduction to the notion of ideal convergence which also includes some elementary properties and examples of ideal convergence.

In **chapter second**, we introduce I-convergent sequence spaces $\mathcal{S}^I(f)$, $\mathcal{S}_0^I(f)$ and $\mathcal{S}_\infty^I(f)$ with the help of compact operator T on the real space $\mathbb{R}$ and a modulus function f. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

In **Chapter third** we introduce the sequence spaces ${}_0BV_\sigma^I(M)$, $BV_\sigma^I(M)$ and ${}_\infty BV_\sigma^I(M)$ with the help of BV_σ space already introduced by Mursaleen [85] and an Orlicz function M. We study some topological and algebraic properties and decomposition theorem. Further we prove some inclusion relations related to these new spaces.

In **chapter forth**, we introduce and study sequence spaces $\mathcal{C}^I(\mathcal{T}, f, p)$, $\mathcal{C}_0^I(\mathcal{T}, f, p)$ and $B_\infty^I(\mathcal{T}, f, p)$ on the sequence of bounded linear operators with the help of a modulus functions f and a bounded sequence $p = (p_k)$ of positive real numbers. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

In **chapter fifth**, we introduce and study I-convergent sequence spaces $\mathcal{S}^I(M)$, $\mathcal{S}_0^I(M)$ and $\mathcal{S}_\infty^I(M)$ with the help of compact operator T on the real space $\mathbb{R}$ and an Orlicz function M. We study some topological and algebraic properties and prove some inclusion relations on these spaces.

In **chapter sixth**, we introduce the intuitionistic fuzzy Zweier I-convergent sequence spaces $\mathcal{Z}_{(\mu,\nu)}^I$ and $\mathcal{Z}_{0(\mu,\nu)}^I$ and study the fuzzy topology on the said spaces.

The thesis ends with a fairly exhaustive bibliography of books and research articles consulted for the work.

CHAPTER 1

PRELIMINARIES

CHAPTER-1

PRELIMINARIES

1.1. NOTATIONS AND BASIC SEQUENCE SPACES:

The primary aim of this section is to recall some basic notations and the definitions of some sequence spaces that we have used and applied freely and frequently in the subsequent work for our own convenience.

1.1.1. NOTATIONS

$\mathbb{N}$:= The set of all natural numbers.

$\mathbb{R}$:= The set of all real numbers.

$\mathbb{C}$:= The set of all complex numbers.

$\lim\limits_k$: means $\lim\limits_{k\to\infty}$.

$\sup\limits_k$: means $\sup\limits_{k\geq 1}$, unless and otherwise stated.

$\inf\limits_k$: means $\inf\limits_{k\geq 1}$, unless and otherwise stated.

$\sum\limits_k$: means summation over $k = 1$ to $k = \infty$, unless and otherwise stated.

$x := (x_k)$, the sequence whose k^{th} term is x_k.

$\theta := (0, 0, 0, ...)$, the zero sequence.

$e_k := (0, 0, .., 1, 0, 0, ..)$, the sequence whose k^{th} component is 1 and others are zeroes, for all $k \in \mathbb{N}$.

$e := (1, 1, 1, 1, ...)$.

''As are the crests on the heads of peacocks, as are the gems on the hoods of cobras, so is Mathematics, at the top of all sciences." -*The Yajurveda, Circa 600 B.C.*

$p := (p_k)$, the sequence of strictly positive reals, unless and otherwise stated.

1.1.2. SOME BASIC SEQUENCE SPACES

Here we give some popular sequence spaces which were introduced and discussed by Simons [100], Maddox [70], [71], Kamthan and Gupta [28] and the references therein and frequently used by various authors that remained instrumental for our successive work .

$\omega := \{x = (x_k) \in \omega : x_k \in \mathbb{R} \quad (\text{or} \quad \mathbb{C})\}$, the space of all sequences, real or complex.

$\ell_\infty := \{x = (x_k) \in \omega : \sup\limits_k |x_k| < \infty\}$, the space of bounded sequences.

$c_0 := \{x = (x_k) \in \omega : \lim\limits_k |x_k| = 0\}$, the space of null sequences.

$c_{00} :=$ The space of finite sequences, i.e. of all sequences terminating in zeros.

$c := \{x = (x_k) \in \omega : \lim\limits_k x_k = L, \quad \text{for some} \quad L \in \mathbb{C}\}$, the space of convergent sequences.

$\zeta := \{x \in \omega : \sup\limits_k |x^n - x^m| \to 0, \ as\ n, m \to \infty\}$, the space of all Cauchy sequences

$\ell_1 := \{x = (x_k) \in \omega : \sum\limits_k |x_k| < \infty\}$, the space of absolutely convergent series.

$w_\infty := \{x = (x_k) \in \omega : \sup\limits_n \frac{1}{n} \sum\limits_k |x_k| < \infty\}$, the space of strongly Cesàro-bounded sequences.

$w_0 := \{x = (x_k) \in \omega : \lim\limits_n \frac{1}{n} \sum\limits_k |x_k| = 0\}$, the space of strongly Cesàro-null sequences.

$\ell_p := \{x = (x_k) \in \omega : \sum\limits_k |x_k|^p < \infty\}$, $0 < p < \infty$.

$w_p := \{x = (x_k) \in \omega : \lim\limits_n \frac{1}{n} \sum\limits_k |x_k - L|^p = 0; \quad \text{for some} \quad L \in \mathbb{C}\}$.

$\ell(p) := \{x = (x_k) \in \omega : \sum\limits_k |x_k|^{p_k} < \infty\}$.

$\ell_\infty(p) := \{x = (x_k) \in \omega : \sup\limits_k |x_k|^{p_k} < \infty\}$.

$c(p) := \{x = (x_k) \in \omega : \lim_k |x_k - L|^{p_k} = 0, \text{ for some } L \in \mathbb{C}\}.$

$c_0(p) := \{x = (x_k) \in \omega : \lim_k |x_k|^{p_k} = 0\}.$

$w_\infty(p) := \{x \in \omega : \sup_k (\frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k}) < \infty\}.$

$w(p) := \{x = (x_k) \in \omega : \lim_n (\frac{1}{n} \sum_{k=1}^{n} |x_k - L|^{p_k}) = 0, \text{ for some } L \in \mathbb{C}\}.$

$w_0(p) := \{x = (x_k) \in \omega : \lim_n (\frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k}) = 0\}.$

1.2. SOME DEFINITIONS AND EXAMPLES

In this section we give some instrumental definitions with suitable examples followed by some useful remarks that help us for carrying out further work in the subsequent chapter.

Definition 1.2.1. Let K be a non-trivial scalar valued field and X be a vector space over K. Then a real valued mapping $\| \cdot \|$ on X is said to be a norm on or over X, if it satisfies the following properties.

(i) $\| x \| \geq 0, \ \| x \| = 0 \Leftrightarrow x = 0,$

(ii) $\| \alpha x \| = | \alpha | \| x \|,$

(iii) $\| x + y \| \leq \| x \| + \| y \|$, for all $\alpha \in K;\ x,\ y \in X$.

The pair $(X, \| \cdot \|)$ is called a normed linear space over K.

Remark 1.2.1. If $K = \mathbb{R}$ (field of reals) or $K = \mathbb{C}$ (field of complex), then X is called a real/complex normed linear space, respectively.

A normed linear space X is said to be a Banach space, if it is complete. That is, if every Cauchy sequence in X is convergent in X.

Example 1.2.1. The spaces ℓ_∞, c and c_0 are the Banach spaces of bounded, convergent and null sequences, respectively with norm

$$\|x\| = \sup_k | x_k | .$$

Definition 1.2.2.[97] A linear metric space (X, d) is a linear space X with a translation invariant metric d on X such that addition and scalar multiplication are continuous in (X, d).

Definition 1.2.3.[97], [107] A paranorm is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $x,\ y \in X$,

(i) $g(x) = 0$ if $x = \theta$,

(ii) $g(-x) = g(x)$,

(iii) $g(x + y) \leq g(x) + g(y)$,

(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X and the pair (X, g) is called a totally paranormed space. If we suppose that (X, d) is a linear metric space and for each $x \in X$ let us define $g(x) = d(x, \theta)$ where θ is the zero element in X, then it is straightforward to check that the properties of d imply the above properties of g.

Definition 1.2.4.[97], [107] A seminorm is a function $v : X \to \mathbb{R}$, defined on a linear space X such that for all $x,\ y \in X$,

(i) $v(x) = 0$, if$x = \theta$,

(ii) $v(\alpha x) =| \alpha | v(x)$, for all scalar α,

(iii) $v(x + y) \leq v(x) + v(y)$.

The property expressed by (ii) is called absolute homogeneity of v and that expressed by (iii) is called subadditivity of v. Thus, a seminorm is a real valued subadditive and absolute homogeneous function on a linear space X.

Moreover, it follows from (ii) and (iii) that

$$0 = v(\theta) = v(x + (-x)) \leq v(x) + v(-x) = 2v(x), \quad \text{for all x.}$$

Whence a seminorm is necessarily non-negative.

Example 1.2.2. For each $x \in \mathbb{C}$, $v(x) =| x |$ defines a seminorm on $\mathbb{C}$.

Example 1.2.3. The mappings $v_1 : c \to \mathbb{R}$ and $v_2 : c \to \mathbb{R}$ defined by $v_1(x) = \sup_k | x_k |$ and $v_2(x) =| \lim_k |$, respectively on the linear space c of all convergent subsequences $x = (x_k)$ are seminorm on c.

Example 1.2.4. Let $p = (p_k)$ be bounded. Then, $c_0(p)$ is a linear metric space paranormed by:

$$g(x) = \sup_k |x_k|^{\frac{p_k}{M}},$$

where $M = \max(1, \sup_k p_k)$.

Example 1.2.5. Let $p = (p_k)$ be bounded. Then, $\ell_\infty(p)$ and $c(p)$ are paranormed spaces, paranormed by $g(x)$ defined above, if and only if $\inf_k p_k > 0$.

Example 1.2.6. Let $p = (p_k)$ be bounded. Then, $\ell(p)$ and $w(p)$ are paranormed spaces, paranormed by

$$g_2(x) = \left(\sum_k |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Example 1.2.7. Let $p = (p_k)$ be bounded. Then, $\ell(p)$ is a totally paranormed space.

Remark 1.1.2. If $p_k = p$, for all k, then $\ell_\infty(p) = \ell_\infty$, $c_0(p) = c_0$, $c(p) = c$, $\ell(p) = \ell$ and $w(p) = w_p$.

Remark 1.2.3. If (X, g) is a totally paranormed space , then it follows readily that d defined by $d(x, y) = g(x - y)$ is such that (X, d) is a linear metric space .

Example 1.2.8. In the case $1 \leq p < \infty$, the space ℓ_p and w_p are Banach spaces, normed by

$$\|x\| = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$$

and

$$\|x\| = \sup\left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}},$$

respectively.

Example 1.2.9. If $0 < p < 1$, then l_p and w_p are complete p-normed spaces, p-normed by

$$\|x\| = \sum_k |x_k|^p$$

and

$$\|x\| = \frac{1}{n}\sum_{k}^{n} |x_k|^p,$$

respectively.

Remark 1.2.4. A linear metric space and a totally paranormed space are really the same thing, likewise linear semimetric space and a paranormed space are equivalent.

Remark 1.2.5.[107] Every seminorm space is paranorm but not conversely.

Example 1.2.10. Consider the paranorm g on ℓ_p, where

$$g(x) = \sum | x_k |^{p_k}$$

and $p_k = \frac{1}{k}$, for all $k \in \mathbb{N}$. Let $x = (0, 1, 0, 0, ...)$. Then $g(2x) \leq 2g(x)$. Thus, g is not a seminorm.

Definition 1.2.5.[97], [90] Let ω be a linear space of sequences. Then, any subspace λ of ω is called a sequence space. A sequence space λ with linear topology is called a K -space, if each of the maps $P_k : X \to \mathbb{C}$ defined by $P_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$. A K -space λ is called an FK-space, if λ is complete linear metric space. In other words, we say that λ is an FK-space, if λ is Fréchet space with continuous coordinate projection, we mean, if $x^{(n)} \to x$ $(n \to \infty)$ in the metric of λ, then $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$, the linear functional $P_k(x) = x_k$ is such that P_k is continuous on λ. An FK-space whose topology is normable is called a BK-space.

Example 1.2.11. The space ω is FK-space with its usual metric.

Remark 1.2.6. A BK-space is a normed FK-space.

Definition 1.2.6.[79] A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus function, if

(i) $f(t) = 0$ if and only if $t = 0$,

(ii) $f(t+u) \leq f(t) + f(u)$ for all t, u$\geq$0,

(iii) f is increasing, and

(iv) f is continuous from the right at zero.

A modulus function f is said to satisfy $\Delta_2 -$ Condition for all values of u, if there exists a constant $K > 0$ such that $f(Lu) \leq \mathrm{KL} f(u)$ for all values of $L > 1$.

Remark 1.2.7. For any modulus function f, we have the inequalities

(1) $\mid f(x) - f(y) \mid \leq f(\mid x - y \mid)$,

(2) $f(nx) \leq nf(x)$, for all $x,\ y \in [0, \infty]$.

Example 1.2.12. Let $f : [0, \infty) \longrightarrow [0, \infty)$ be defined as $f(x) = x^p$ $(0 < p \leq 1)$ for all $x \in [0, 1)$. Then, f is a modulus function.

Definition 1.2.7.[68] A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called convex, if

$$M\left(\frac{u+v}{2}\right) \leq \frac{M(u) + M(v)}{2}, \qquad \text{for all } u, v \in \mathbb{R}.$$

If in addition, the two sides of above are not equal for $u \neq v$, then we call M to be strictly convex.

Example 1.2.13. Let $M : [0,\infty) \longrightarrow [0,\infty)$ be defined as $M(x) = x^2$ for all $x \in [0, 1)$. Then, M is a convex function.

Definition 1.2.8.[68], [108] A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function, if

(i) M is continuous,

(ii) M is nondecreasing,

(iii) M is convex,

(iv) $M(0) = 0$, $M(x) > 0$ for $x > 0$,

(v) $M(x) \to \infty$ as $x \to \infty$.

If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it becomes a Modulus function, defined and discussed by Nakano [79], Ruckle [92], Bhardwaj [6], [7] and many others.

An Orlicz function M is said to satisfy the $\Delta_2 -$ *condition* ($M \in \Delta_2$ for short), if there exist constant $K \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Lindenstrauss and Tzafriri [68] used the idea of Orlicz function to construct the sequence space;

$$\ell_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M becomes a Banach space with the norm

$$\| x \| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{| x_k |}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = t^P$ for $1 < p < \infty$.
which is a Banach space with the norm

$$\|x\|_M = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Example 1.2.14. Let $M : [0,\infty) \longrightarrow [0,\infty)$ be defined as $M(x) = \frac{x}{x+1}$ for all $x \in [0, 1)$. Then, M is an Orlicz function.

Remark 1.2.8. An Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

Further study of Orlicz function and its related sequence spaces was done by Yan [108], Et [15], Esi [3], Bhardwaj and Singh [6], Parashar and Choudhary [87], Musraleen and Sunil [78], Tripathy and Hazarika [101], Khan [48], [49], [53], Khan and Tabassum [55]-[59], Khan and Ebadullah [50], Khan *et al* [53] and many others.

Definition 1.2.9.[95] Let X be any non-empty space. A map $\hbar : D \subset X \rightarrow \mathbb{R}$ defined on a domain $D \subset X$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by (D, K).

Example 1.2.15. The function $f(x) = | x |$ defined for all real numbers is Lipschitz continuous with the Lipschitz constant $K = 1$. The function $f(x) = x^2$ defined on defined on all real numbers is not Lipschitz continuous.

Definition 1.2.10.[18] Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then, for any sequence space E, the Multiplier Sequence $E(\Lambda)$ of E, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

Definition 1.2.11.[101] Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of pre-images all elements in λ_K^E. That is, y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

A sequence space E

(i) is said to be solid(normal), if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

(ii) is said to be symmetric, if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$.

(iii) is said to be sequence algebra, if $(x_k) * (y_k) = (x_k . y_k) \in E$ whenever $(x_k), (y_k) \in E$.

(iv) is said to be convergence free, if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

(v) is said to be monotone, if it contains the canonical preimages of its step space.

Remark 1.2.9.[28] Every solid space is monotone.

Definition 1.2.12.[4], [77] A Banach limit L is a linear function on ℓ_∞ such that

(i) $L(x) \geq 0$, if $x_k \geq 0, k \geq 0$,

(ii) $L(Dx) = L(x)$ for all $x \in \ell_\infty$,

(iii) $L(e) = 1$, where $e = \{1, 1, 1...\}$.

where D is a shift operator on S (set of all sequences of real or complex numbers), $D\{x_n\} = \{x_{n+1}\}$.

Definition 1.2.13.[76] Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an Invariant Mean or σ-mean, if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k

(ii) $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$ and

(iii) $\phi(x_{\sigma(k)}) = \phi(x)$ for all $x \in \ell_\infty$.

Definition 1.2.14. Let ℓ_∞ be denote the linear space of bounded sequences. A sequence $x = (x_k) \in \ell_\infty$ is said to be Almost Convergent and s ia called its generalised limit, if each Banach limits [4] of x is coincide and equal to s. Let $\hat{c}$, $\hat{c}_0$ be denote the set of sequences which are almost convergent and almost convergent to zero, respectively.

Lorentz [[69], Theorem 1] proved that $x = (x_k) \in \hat{c}$, if and only if

$$\lim_{p\to\infty} \frac{x_n + x_{n+1} + ... + x_{n+p-1}}{p} = s$$

uniformly in n.

A convergent sequence is almost convergent and its limit and its generalised limit are identical. Lorentz [69] also proved that $\hat{c}$ is a closed linear subspace of ℓ_∞ in the usual topology.

1.3. STATISTICAL CONVERGENCE

Statistical convergence is a generalization of the usual notion of convergence. The idea of statistical convergence was given in the first edition (published in Warsaw in 1935) of the monograph of Zygmund [110] where he called it "almost convergence". Formally the concept of statistical convergence was introduced by Fast [16] in the year 1951 and reintroduced by Buck [8] in 1953 and Schoenberg [98] in the year

1959 for real and complex sequences. Although statistical convergence was introduced over nearly last sixty years back, it has become an active area of research in recent years. This concept has been studied and applied by various researchers in different areas of mathematics such as Erdös and Tenenbaum [13] in number theory, Miller [75] in measure theory , Zygmund [110] in trigonometric series, Gadjier and Orhan [20], and Duman *et al* [12] in approximation theory, Freedman *et al* [17] in summability theory and many others. Most recently, this concept has been studied for sequence space point of view by Fridy [18], [19], Cannor [9],[10], Šalát [94], Tripathy [103], Khan and Tabassum [40] and many others.

Definition 1.3.1.[63] If $A \subseteq \mathbb{N}$, then χ_A denotes characteristic function of the set A. That is,

$$\chi_A(k) = \begin{cases} 1, & \text{if } k \in A, \\ 0, & \text{if } k \in \mathbb{N} \setminus A. \end{cases}$$

Put

$$d_n(A) = \frac{1}{n} \sum_{k=1}^{n} \chi_A(k),$$

$$\delta_n(A) = \frac{1}{S_n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k} \quad (n = 1, 2, 3, ...),$$

where

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Then, the numbers

$$\underline{d}(A) = \liminf_{n \to \infty} d_n(A),$$

$$\bar{d}(A) = \limsup_{n \to \infty} d_n(A),$$

are called the lower and upper asymptotic density of A, respectively. Similarly, the numbers

$$\underline{\delta}(A) = \liminf_{n \to \infty} \delta_n(A),$$

$$\bar{\delta}(A) = \limsup_{n \to \infty} \delta_n(A),$$

are called the lower and upper logarithmic density of A, respectively. If there exists

$$\lim_{n \to \infty} d_n(A) = d(A),$$

and

$$\lim_{n\to\infty} \delta_n(A) = \delta(A),$$

then $d(A)$ and $\delta(A)$ are called the asymptotic and logarithmic density of A, respectively. It is well known fact, that for each $A \subseteq \mathbb{N}$,

$$\underline{d}(A) \leq \underline{\delta}(A) \leq \bar{\delta}(A) \leq \bar{d}(A).$$

Hence, if $d(A)$ exists, then $\delta(A)$ exists as well and $d(A) = \delta(A)$. The numbers $\underline{d}(A), \bar{d}(A), d(A), \underline{\delta}(A), \bar{\delta}(A), \delta(A)$ belong to the interval [0,1].

Statistical convergence depends on the density of the set $\mathbb{N}$ of natural numbers. The natural density [86] of a subset $K \subseteq \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} n^{-1} \mid \{k \in K : k \leq n\} \mid,$$

whenever the limits exists where $\mid \{k \in K : k \leq n\} \mid$ denotes the number of elements of K not exceeding n.

Let us understand this concept of natural density with the help of some examples.

Example 1.3.1. Let $K = \{1, 2, 3, ...\} = \mathbb{N}$. Then, $\mid \{k \in K : k \leq n\} \mid = n$ and so $\lim_{n\to\infty} n^{-1} \mid \{k \in K : k \leq n\} \mid = 1$. Thus, $\delta(K) = \delta(N) = 1$. That is, the natural density of the set of natural numbers is one.

Example 1.3.2. Let $K = \{2, 4, 6, ...\}$, the set of all even natural numbers. Then,

$$\mid \{k \in K : k \leq 1\} \mid = 0,$$

$$\mid \{k \in K : k \leq 2\} \mid = 1,$$

$$\mid \{k \in K : k \leq 3\} \mid = 1,$$

$$\mid \{k \in K : k \leq 4\} \mid = 2,$$

$$\mid \{k \in K : k \leq 5\} \mid = 2,$$

$$\mid \{k \in K : k \leq 6\} \mid = 3,$$

$$\mid \{k \in K : k \leq 7\} \mid = 3, \text{ and so on.}$$

That is,

$$|\{k \in K : k \leq n\}| = \begin{cases} \frac{n}{2}, & \text{if n is even,} \\ \frac{n-1}{2}, & \text{if n is odd.} \end{cases}$$

Therefore,

$$\frac{1}{n} |\{k \in K : k \leq n\}| = \begin{cases} \frac{1}{2}, & \text{if n is even,} \\ \frac{1}{2} - \frac{1}{2n}, & \text{if n is odd.} \end{cases}$$

and so

$$\lim_{n\to\infty} n^{-1} |\{k \in K : k \leq n\}| = \frac{1}{2}.$$

Thus, $\delta(K) = \frac{1}{2}$. That is, the set of all even natural numbers has $\frac{1}{2}$ natural density.

Example 1.3.3. Let $K = \{k : k = n^2,\ n \in \mathbb{N}\}$, the set of squares of natural numbers. Then,

$$|\{k \in K : k \leq 1\}| = 1 \leq \sqrt{1},$$

$$|\{k \in K : k \leq 2\}| = 1 \leq \sqrt{2},$$

$$|\{k \in K : k \leq 3\}| = 1 \leq \sqrt{3},$$

$$|\{k \in K : k \leq 4\}| = 2 \leq \sqrt{4},$$

$$|\{k \in K : k \leq 5\}| = 2 \leq \sqrt{5},$$

$$|\{k \in K : k \leq 6\}| = 2 \leq \sqrt{6}, \text{ and so on.}$$

That is,

$$|\{k \in K : k \leq n\}| \leq \sqrt{n} \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} n^{-1} |\{k \in K : k \leq n\}| \leq \lim_{n\to\infty} n^{-1}\sqrt{n} = 0.$$

Thus, the set of squares of natural numbers has zero natural density.

Remark 1.3.1. Every finite subset of $\mathbb{N}$ has natural density zero.

Remark 1.3.2. Let $K = \{1.3, 5., ...\}$, the set of all odd natural numbers. Then, $\delta(K) = \frac{1}{2}$.

Remark 1.3.3. Let K be any subset of $\mathbb{N}$. Then, $\delta(K^c) = 1 - \delta(K)$.

Definition 1.3.2.[18], [94] A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$, if for every $\varepsilon > 0$, we have

$$\lim_{k\to\infty} \frac{1}{k}|\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon, n \leq k\}| = 0,$$

where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\varepsilon)) = 0$, where

$$A(\epsilon) = \left\{k \in \mathbb{N} : | x_k - L | \geq \varepsilon\right\}$$

In this case, following Fridy [18], we write

$$st - \lim x_k = L.$$

Instead of this notation, Šalát [94] used the notation

$$\lim_{k\to\infty} stat\ x_k = L.$$

whereas Schoenberg [98] used the notation

$$D - \lim x_k = L.$$

and called statistical convergence as D-convergence.

Remark 1.3.4. In usual convergence it is required that for every $\epsilon > 0$, the set $\{k \in \mathbb{N} :| x_k - L | \geq \varepsilon\}$ should be finite whereas in statistical convergence, this set should have natural density zero.

Remark 1.3.5. A sequence which converges statistically need not be convergent.

Here we give some examples of statistical convergence.

Example 1.3.4. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k = n^2, \quad n \in \mathbb{N}, \\ 0, & \text{otherwise} . \end{cases}$$

That is,

$$x = \{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, ...\}$$

and let $L = 0$. Then,

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \subset \{1, 4, 9, 16, ..., j^2, ...\}.$$

We have that

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0, \quad \text{for every} \quad \epsilon > 0.$$

This implies that the sequence (x_k) converges statistically to zero. But the sequence (x_k) does not converge to $L = 0$.

Remark 1.3.6. A sequence which converges statistically need not be bounded.

Here we give a counter example.

Example 1.3.5. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} k, & \text{if } k = n^2, \quad n \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

That is,

$$x = \{1, 0, 0, 4, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 16, 0, 0, ...\}$$

Then, this sequence is statistical convergent but not bounded.

Definition 1.3.3.[104] A sequence x=$(x_k) \in \omega$ is said to be statistically bounbed, if there exists a number $M > 0$ such that

$$\delta(\{n \in \mathbb{N} : |x_k| \geq M\}) = 0.$$

That is, $|x_k| \leq M$ a.a.k.

Remark 1.3.7. Every bounded sequence is statistical bounded and every statistical convergent sequence is statistical bounded but their converses are not true.

Here we give some counter examples.

Example 1.3.6. The above example (1.3.5.) is statistically bounded but not bounded.

Example 1.3.7. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} \sqrt{k}, & \text{if } k = n^2, \quad n \in \mathbb{N}, \\ 0, & \text{if k is odd non square,} \\ 1, & \text{if k is even non square.} \end{cases}$$

Then, this sequence is statistically bounded but not statistically convergent.

Example 1.3.8. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} k, & \text{if k is an odd square,} \\ 2, & \text{if k is an even square,} \\ 1, & \text{if k is odd non square,} \\ 0, & \text{if k is even non square .} \end{cases}$$

Then, this sequence is statistically bounded because the set of squares has density zero. However, it not statistically convergent, since it has two (disjoint) subsequences of positive density that converge to 0 and 1, respectively.

1.4. IDEAL CONVERGENCE

The notion of ideal convergence (I-convergence) of sequences was introduced and studied by Kostyrko, Salăt and Wilczyński [63] in the year 2000. It was further studied by Kostyrko, Mačaj, Salăt and Sileziak [64] for extremal I-limit points in the year 2005. Further research and contributions in this direction were done by Šalát, Tripathy and Ziman [95], [96], Gürdal and Acik [23], Gürdal and Sahiner [24], Mursaleen and Sunil [78], Demirci [11], Esi and Hazarika [2], Hazarika and Esi [26], Hazarika *et al* [27], Tripathy and Hazarika [101], [102], [105], Khan *et al* [42]-[62] ; and many others.

Definition 1.4.1.[64] Let N be a non empty set. Then, a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal, if

(i) I is additive. That is, $\forall A, B \in I \Rightarrow A \cup B \in I$

(ii) I is hereditary. That is, $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 1.4.2.[64] A non-empty family of sets $\mathcal{J} \subseteq 2^N$ is said to be filter on N, if and only if

(i) $\Phi \notin \mathcal{J}$,

(ii) $\forall\, A, B \in \mathcal{J}$ we have $A \cap B \in \mathcal{J}$,

(iii) $\forall\, A \in \mathcal{J}$ and $A \subseteq B \Rightarrow B \in \mathcal{J}$.

Definition 1.4.3. An Ideal $I \subseteq 2^N$ is called non-trivial, if $I \neq 2^N$.

Definition 1.4.4 A non-trivial ideal $I \subseteq 2^N$ is called admissible, if

$$\{\{x\} : x \in N\} \subseteq I.$$

Definition 1.4.5. A non-trivial ideal I is maximal, if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 1.4.1. For each ideal I, there is a filter $\mathcal{L}(I)$ corresponding to I. i.e $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Definition 1.4.6.[64] A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L, if for every $\epsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I.$$

In this case, we write $I - \lim x_k = L$.

Definition 1.4.7.[64] A sequence $x = (x_k) \in \omega$ is said to be I-null, if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.4.8.[101] A sequence $x = (x_k) \in \omega$ is said to be I-Cauchy, if for every $\epsilon > 0$, there exists a number $m = m(\epsilon)$ such that the set

$$\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I.$$

Definition 1.4.9.[101] A sequence $x = (x_k) \in \omega$ is said to be I-bounded, if there exists some $M > 0$ such that the set

$$\{k \in \mathbb{N} : |x_k| \geq M\} \in I.$$

Definition 1.4.10.[102] Let $x = (x_k)$ and $y = (y_k)$ be two sequences. We say that $x_k = y_k$ for almost all k relative to I (a.a.k.r.I), if the set

$$\{k \in \mathbb{N} : x_k \neq y_k\} \in I.$$

Remark 1.4.2.[65], [102] Let $K \in \mathcal{L}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

Remark 1.4.3.[102] If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$.

Definition 1.4.11. [95], [96] A convergence field of I-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ [95], [96] is a closed linear subspace of ℓ_∞ with respect to the supremum norm and $F(I) = \ell_\infty \cap c^I$.

Remark 1.4.4. Let I be an admissible ideal in $\mathbb{N}$. Then $F(I) = \ell_\infty$ if and only if I is a maximal admissible ideal in $\mathbb{N}$.

Define a function [95] $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function.

Here we give some examples of Ideal.

Example 1.4.1. Let $I_0 = \{\emptyset\}$. This is the minimal non-empty non-trivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is I_0- convergent if and only if it is constant.

Example 1.4.2. Let $\emptyset \neq M \subseteq \mathbb{N}$, $M \neq \mathbb{N}$. Let $I_M = 2^M$. Then I_M is a non trivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is I_M-convergent if and only if it is constant on $\mathbb{N} \backslash M$.

Example 1.4.3. Let I_f denotes the class of all finite subsets of $\mathbb{N}$. Then I_f is an admissible ideal in $\mathbb{N}$ and I_f-convergence coincides with the usual convergence in $\mathbb{R}$.

Example 1.4.4. Let $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then I_d is an admissible ideal in $\mathbb{N}$ and I_d-convergence coincides with the statistical convergence.

Example 1.4.5. Let $I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I_δ is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence the logarithmic statistical convergence.

Example 1.4.6. Let $u(A)$ denotes the uniform density of the set A. Then $I_u = \{A \subseteq \mathbb{N} : u(A) = 0\}$, is an admissible ideal in $\mathbb{N}$ and I_u-convergence will be called the uniform statistical convergence.

Remark 1.4.5. If $I_\delta - \lim x_k = L$, then $I_d - \lim x_k = L$

Comparison between usual convergence and I-convergence.

A question arises whether the concept of I-convergence satisfies some usual axioms of convergence. The most known axioms of convergence are the following axioms (formulated for I-convergence).

(A) Every stationary sequence $x = (c, c, ...c, ...)$ I-converges to c.

(B) The uniqueness of limit: If $I - \lim x = L_1$ and $I - \lim x = L_2$, then $L_1 = L_2$.

(C) If $I - \lim x = L$, then for each subsequence y of x we have $I - \lim y = L$.

(D) If each subsequence y of a sequence x has a subsequence z I-convergent to L, then x is I-convergent to L.

Theorem 1.4.1.[64], [65] Let I be an admissible ideal in $\mathbb{N}$. Then

(i) I-convergence satisfies the axioms (A), (B) and (D).

(ii) if I contains an infinite set, then I-convergence does not satisfy the axioms (C).

Remark 1.4.6. If an admissible ideal contains no infinite set, then I coincides with the class of all finite sunsets of $\mathbb{N}$ and the I-convergence is equal to the usual convergence in $\mathbb{R}$, therefore it satisfies the axiom (C).

Theorem 1.4.2.[64], [65] Let I be a non-trivial ideal of $\mathbb{N}$

(i) If $I - \lim x_k = L_1$, $I - \lim y_k = L_2$, then $I - \lim(x_k + y_k) = L_1 + L_2$.

(ii) If $I - \lim x_k = L_1$, $I - \lim y_k = L_2$, then $I - \lim(x_k.y_k) = L_1.L_2$.

(iii) If I is an admissible ideal in $\mathbb{N}$, then $\lim\limits_{k\to\infty} x_k = L$ implies $I - \lim x_k = L$.

Theorem 1.4.3.[65] Let $I \subset 2^{\mathbb{N}}$ be a maximal admissible ideal. Then, each bounded sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is I-convergent. That is, there exists $L \in \mathbb{R}$ such that $I - \lim x_k = L$.

Theorem 1.4.4.[65] Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then, there exists an unbounded sequence $x = (x_k)$ of real numbers for which $I - \lim x_k$ does not exist.

Example.1.4.7. Consider the sequence $x_k = k$, k=1,2,3,.... Then, it is a wanted one example.

Functions Preserving I-Convergence.

Definition 1.4.12.[65] Let (X, ρ) be metric space and $I \subset 2^N$ an admissible ideal. Then, we say that a function $g : X \to X$ <u>preserves I-convergence</u> in X, if $I - \lim\limits_{k\to\infty} x_k = L$ implies that $I - \lim\limits_{k\to\infty} g(x_k) = g(L)$ for each sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements of X and each $L \in X$.

Theorem 1.4.5.[65] Let I be an admissible ideal. A function $g : X \to X$ preserves I-convergence in X if and only if g is continuous on X.

I-Convergence of Sequence of Functions.

In a natural manner we can extend the notation of I-convergence of sequence in X to I-convergence of sequence of functions.

Definition 1.4.13.[65] Let X be a non-empty set and let (Y, τ) be a metric space. Let $I \subset 2^N$ be an admissible ideal.. Then, the sequence of functions $f_k (k \in \mathbb{N})$ transforming X into Y is said to <u>I-converges</u> to a function $f : X \to Y$ provided that for each $x \in X$, we have $I - \lim\limits_{k\to\infty} f_k(x) = f(x)$.
The function f is called the I-limit function of the sequence $f_k (k \in \mathbb{N})$ and we write it as $I - \lim\limits_{k\to\infty} f_k = f$

Remark 1.4.7. If I is a maximal admissible ideal, then for each $A \subset \mathbb{N}$, we have either $A \in I$ or $\mathbb{N} \setminus A \in I$.

Remark 1.4.8. The spaces of all I-null, I-convergent, I-bounbed sequences are denoted by c_0^I, c^I, ℓ_∞^I, respectively.

1.5. SOME INEQUALITIES.

Here,we give some inequalities which will help us in the subsequent work.

1.5.1. Triangular Inequality.[71] Let $\alpha,\ \beta \in \mathbb{C}$, then

$$|\alpha+\beta| \leq |\alpha| + |\beta|$$

1.5.2. Let $\alpha,\ \beta \in \mathbb{C}$, then

$$\frac{|\alpha+\beta|}{1+|\alpha+\beta|} \leq \frac{|\alpha}{1+|\alpha|} + \frac{|\beta|}{1+|\beta|}.$$

1.5.3. Let $0 < \lambda < 1$, then

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta,$$

for every pair of non-negative real numbers $\alpha,\ \beta$ with equality holds only if $\alpha = \beta$.

1.5.4. Hölders Inequality.[71] Let $1 < p,\ q < \infty$ and p and q be conjugate exponents.

If $\alpha_i, \beta_i \in \mathbb{C}$ $(i = 1, 2, 3, ..., n)$, then

$$\sum_{i=1}^{n} |\alpha_i \beta_i| \leq \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\beta_i|^q\right)^{\frac{1}{q}}.$$

Also,

$$\sum_{i=1}^{n} |\alpha_i \beta_i| \leq \sum_{i=1}^{n} |\alpha_i|^p \max_{1\leq i\leq n} |\beta_i|.$$

1.5.5. Minkowski inequality Inequality.[71] Let $1 \leq p < \infty$.

If $\alpha_i,\ \beta_i \in \mathbb{C}(i = 1, 2, 3, ..., n)$, then

$$\left(\sum_{i=1}^{n} \mid \alpha_i + \beta_i \mid^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \mid \alpha_i \mid^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \mid \beta_i \mid^p\right)^{\frac{1}{p}}.$$

Also, Let $0 < p \leq 1$. If $\alpha_i,\ \beta_i \in \mathbb{C}\ (i = 1, 2, 3, ..., n)$, then

$$\sum_{i=1}^{n} \mid \alpha_i + \beta_i \mid^p \leq \sum_{i=1}^{n} \mid \alpha_i \mid^p + \sum_{i=1}^{n} \mid \beta_i \mid^p .$$

Inequalities (1.5.1) and (1.5.5) yield the following frequently used results valid for complex numbers α_i, β_i

$$(\sum |\alpha_i + \beta_i|^p)^{1/p} \leq (\sum(|\alpha_i|^p)^{1/p} + (\sum(|\beta_i|^p)^{1/p}, \quad (p \geq 1),$$

$$\sum |\alpha_i + \beta_i|^p \leq \sum |\alpha_i|^p + \sum |\beta_i|^p, \quad (0 < p \leq 1).$$

1.5.6.[70], [71] For any $E > 0$ and any two complex numbers $a,\ b$

(i) $\mid ab \mid \leq E(\mid a \mid^q E^{-q} + \mid b \mid^p), (p > 1, \frac{1}{p} + \frac{1}{q} = 1)$.

(ii) $\mid a \mid^p - \mid b \mid^p \leq \mid a + b \mid^p \leq \mid a \mid^p + \mid b \mid^p,\ (0 < p \leq 1)$.

(iii) $\mid \lambda \mid^p \leq \max(1, \mid \lambda \mid), (0 < p \leq 1)$, where λ is complex scalar and for $p = (p_k)$ a strictly positive real sequence such that $H = \sup p_k < \infty$.

$$\mid \lambda \mid^{p_k} \leq \max(1, \mid \lambda \mid^H).$$

(iv) $\mid a + b \mid^{p_k} \leq M(\mid a \mid^{p_k} + \mid b \mid^{p_k})$, whenever $H = \sup p_k < \infty$, where $M = \max(1, 2^{H-1})$.

CHAPTER 2

ON I-CONVERGENT SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND A MODULUS FUNCTION

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CHAPTER-2

ON I-CONVERGENT SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND A MODULUS FUNCTION

2.1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\},$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 be denote the Banach spaces of bounded, convergent and null sequences of reals, respectively with norm

$$\|x\| = \sup_k \mid x_k \mid .$$

Any subspace λ of ω is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous, for all $i \in \mathbb{N}$. A space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space.

Definition 2.1.1. Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator T is said to be bounded if there exists a positive real k such that

$$\| Tx \| \leq k \| x \|, \text{ for all } x \in \mathcal{D}(T).$$

The set of all bounded linear operators $B(X, Y)$ is a normed linear space normed by

$$\| T \| = \sup_{x \in X, \|x\|=1} \| Tx \|$$

''Number rules the universe." -*Pythagoras* and $\mathcal{B}(X,Y)$ is a Banach space if Y is Banach space.

Definition 2.1.2. Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator) if
1) T is linear,
2) T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X,Y)$ is closed subspace of $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$ is a Banach space if Y is Banach space.

Following Basar and Altay[5] and Sengönül[39], we introduce the sequence spaces $\mathcal{S}$ and $\mathcal{S}_0$ with the help of compact operator T on the real space $\mathbb{R}$ as follows.

$$\mathcal{S} = \{x = (x_k) \in \ell_\infty : T(x) \in c\}$$

and

$$\mathcal{S}_0 = \{x = (x_k) \in \ell_\infty : T(x) \in c_0\}.$$

Definition 2.1.3. A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is increasing, and
(4) f is continuous from the right at zero.

For any modulus function f, we have the inequalities

$$| f(x) - f(y) | \leq f(| x - y |)$$

and

$$f(nx) \leq nf(x), \qquad \text{for all } x,\ y \in [0,\infty]$$

A modulus function f is said to satisfy Δ_2-Condition for all values of u if there exists a constant $K > 0$ such that $f(Lu) \leq \mathrm{KL}f(u)$ for all values of $L > 1$.

The idea of modulus was introduced by Nakano in 1953.(See[79] , Nakano, 1953). Ruckle [90]-[92] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK-space and Ruckle[90]-[92] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[90]-[92] proved that, for any modulus f.

$$X(f) \subset \ell_1 \text{ and } X(f)^{\alpha} = \ell_{\infty}$$

where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch [22]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[21], G.Köthe[66] and W.H.Ruckle[90]-[92].

The sequence spaces by the use of modulus function was further investigated by Maddox[70]-[72], Khan[31],[36], Bhardwaj[7] and many others.

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast[16] and also independently by Buck[8] and Schoenberg[98] for real and complex sequences. Later on, it was further investigated from sequence space point of view and linked with the Summability Theory by Fridy[18], Šalát[94], Tripathy[103], Khan[32], Khan and Sabiha[41],Khan,Shafiq and Rababah[60] and many others.

Definition 2.1.4. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have

$$\lim_{k\to\infty}\frac{1}{k}|\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon, n \leq k\}| = 0.$$

where vertical lines denote the cardinality of the enclosed set.
That is, if $\delta(A(\varepsilon)) = 0$, where

$$A(\epsilon) = \left\{k \in \mathbb{N} :| x_k - L |\geq \varepsilon\right\}$$

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyński[65]. Later on, it was studied by Šalát, Tripathy and Ziman [95],[96], Tripathy and Hazarika[101],[102], Khan and Ebadullah[44], Khan *et.al.*[45] and many others.

Now, we recall the following definitions :

Definition 2.1.5. Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if
1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$
2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 2.1.6. A non-empty family of sets $\mathcal{L}(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if
1) $\Phi \notin \mathcal{L}(\mathrm{I})$,
2) $\forall\ A,\ B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$,
3) $\forall\ A \in \mathcal{L}(I)$ and $A \subseteq B \Rightarrow B \in \mathcal{L}(I)$.

Definition 2.1.7. An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial if $I \neq 2^{\mathbb{N}}$.

Definition 2.1.8. A non-trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if

$$\{\{x\} : x \in \mathbb{N}\} \subseteq I.$$

Definition 2.1.9. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 2.1.10. For each ideal I, there is a filter $\mathcal{L}(I)$ corresponding to I.
i.e $\mathcal{L}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 2.1.11. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 2.1.12. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 2.1.13. A sequence $x = (x_k) \in \omega$ is said to be I-Cauchy if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$.

Definition 2.1.14. A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 2.1.15. A sequence space E is said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$.

Definition 2.1.16. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$

Definition 2.1.17. A sequence space E is said to be sequence algebra if $(x_k) * (y_k) = (x_k . y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 2.1.18. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 2.1.19. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 2.1.20. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 2.1.21. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 2.1.22(see,[60],[65]). If $I = I_f$, the class of all finite subsets of v. Then, I is an admissible ideal in v and I_f convergence coincides with the usual convergence.

Definition 2.1.23(see,[60],[65]). If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 2.1.24. (see,[60],[65]). If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 2.1.25. If $I_\delta - \lim x_k = l$, then $I_d - \lim x_k = l$

Definition 2.1.26. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to I\!\!R$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 2.1.27. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in I\!\!R\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I (see[70], [73])$

Definition 2.1.28. Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x,\ y \in X$,
(P_1) $g(x) = 0$ if $x = \theta$,
(P_2) $g(-x) = g(x)$,
(P_3) $g(x+y) \leq g(x) + g(y)$,
(P_4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

The notation of paranorm sequence spaces was studied at the initial stage by Nakano[79]. Later on, it was further investigated by Maddox[70], Tripathy and Hazarika[102],Khan *et at*[45] and the references therein.

Throughout the article, we use the same techniques as used in [101],[102].

We used the following lemmas for establishing some results of this article.

Lemma(I)(see,[101],[102]). Every solid space is monotone

Lemma(II)(see,[101],[102]). If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

Lemma(III)(see,[101],[102]). Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Throughout the article S^I, S_0^I, S_∞^I, $\mathcal{M}_S^I$ and $\mathcal{M}_{S_0}^I$ represent the I-convergent, I-null, I-Bounded , bounded I-convergent and bounded I-null Sequences spaces defined by a compact operator T on the real space $\mathbb{R}$ respectively.

2.2. MAIN RESULTS

In this article, we introduce the following classes of sequences.

$$\mathcal{S}^I(f) = \left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\Big(\mid T(x_k) - L \mid \Big) \geq \epsilon\} \in I, \text{for some } L \in \mathbb{C} \right\}, \tag{2.2.1}$$

$$\mathcal{S}_0^I(f) = \left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\Big(\mid (Tx_k) \mid \Big) \geq \epsilon\} \in I \right\}, \tag{2.2.2}$$

$$\mathcal{S}_\infty^I(f) = \left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : \exists\, K > 0 \; suchthat \; f\Big(\mid T(x_k) \mid \Big) \geq K\} \in I \right\} \tag{2.2.3}$$

$$\mathcal{S}_\infty(f) = \left\{ x = (x_k) \in \ell_\infty : \sup_k f\big(\mid T(x_k) \mid \big) < \infty \right\} \tag{2.2.4}$$

where f is a modulus function.
We also denote

$$\mathcal{M}_{\mathcal{S}}^I(f) = \mathcal{S}_\infty(f) \cap \mathcal{S}^I(f) \text{ and } \mathcal{M}_{\mathcal{S}_0}^I(f) = \mathcal{S}_\infty(f) \cap \mathcal{S}_0^I(f).$$

Theorem 2.2.1. Let f be a modulus function. Then, the classes of sequences $\mathcal{S}^I(f)$, $\mathcal{S}_0^I(f)$, $\mathcal{M}_{\mathcal{S}}^I(f)$ and $\mathcal{M}_{\mathcal{S}_0}^I(f)$ are linear spaces.

Proof. We shall prove the result for $\mathcal{S}^I(f)$. The proof for the other spaces will follow similarly.
For, let $x = (x_k),\; y = (y_k) \in \mathcal{S}^I(f)$ and α, β be scalars. Then, for a given $\epsilon > 0$, we have

$$\left\{ k \in \mathbb{N} : f\Big(\mid T(x_k) - L_1 \mid \Big) \geq \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \in I \tag{2.2.5}$$

$$\left\{ k \in \mathbb{N} : f\Big(\mid T(x_k) - L_2 \mid \Big) \geq \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \in I \tag{2.2.6}$$

Let

$$A_1 = \left\{ k \in \mathbb{N} : f\Big(|T(x_k) - L_1| \Big) < \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \tag{2.2.7}$$

$$A_2 = \left\{ k \in \mathbb{N} : f\Big(|T(y_k) - L_2| \Big) < \frac{\epsilon}{2}, \text{ for some } L_2 \in \mathbb{C} \right\} \tag{2.2.8}$$

be such that $A_1^c, A_2^c \in I$.
Since f is a modulus function, we have

$$A_3 = \left\{ k \in \mathbb{N} : f\Big(|(\alpha T(x_k) + \beta T(y_k) - (\alpha L_1 + \beta L_2)| \Big) < \epsilon \right\}$$

$$\supseteq \left[\left\{ k \in \mathbb{N} : f\Big(|\alpha||T(x_k) - L_1| \Big) < \frac{\epsilon}{2} \right\} \right.$$

$$\left. \cap \left\{ k \in \mathbb{N} : f\Big(|\beta||T(y_k) - L_2| \Big) < \frac{\epsilon}{2} \right\} \right]$$

$$\supseteq \left[\left\{ k \in \mathbb{N} : f\Big(|T(x_k) - L_1| \Big) < \frac{\epsilon}{2} \right\} \right.$$

$$\left. \cap \left\{ k \in \mathbb{N} : f\Big(|T(y_k) - L_2| \Big) < \frac{\epsilon}{2} \right\} \right]$$

Therefore,

$$A_3 = \left\{ k \in \mathbb{N} : f\Big(|(\alpha T(x_k) + \beta T(y_k) - (\alpha L_1 + \beta L_2)| \Big) < \epsilon \right\}$$

$$\supseteq \left[\left\{ k \in \mathbb{N} : f\Big(|T(x_k) - L_1| \Big) < \frac{\epsilon}{2} \right\} \right.$$

$$\left. \cap \left\{ k \in \mathbb{N} : f\Big(|T(y_k) - L_2| \Big) < \frac{\epsilon}{2} \right\} \right] \tag{2.2.9}$$

implies that $A_3 \in \mathcal{L}(I)$. Thus, $A_3^c = A_1^c \cup A_2^c \in I$. Therefore, $\alpha x_k + \beta y_k \in \mathcal{S}^I(f)$, for all scalars α, β, and (x_k), $(y_k) \in \mathcal{S}^I(f)$.
Hence $\mathcal{S}^I(f)$ is a linear space.

Theorem 2.2.2. The classes of sequences $\mathcal{M}^I_{\mathcal{S}}(f)$ and $\mathcal{M}^I_{\mathcal{S}_o}(f)$ are paranormed spaces, paranormed by

$$g(x) = g(x_k) = \sup_k f\big(|T(x_k)|\big).$$

Proof. Let $x = (x_k)$, $y = (y_k) \in \mathcal{M}^I_{\mathcal{S}}(f)$.
(P_1) It is Clear that $g(x) = 0$ if $x = \theta$, a zero vector.
(P_2) $g(x) = g(-x)$ is obvious.
(P_3) For $x = (x_k)$, $y = (y_k) \in \mathcal{M}^I_{\mathcal{S}}(f)$, we have

$$g(x+y) = g(x_k + y_k) = \sup_k f\big(|T(x_k + y_k)|\big)$$

$$= \sup_k f\big(|T(x_k) + T(y_k)|\big) \le \sup_k f\big(|T(x_k)|\big)$$

$$+ \sup_k f\big(|T(y_k))|\big) = g(x) + g(y).$$

Therefore, $g(x+y) \leq g(x) + g(y)$
(P_4) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda$ $(k \to \infty)$ and (x_k), $L \in \mathcal{M}_S^I(f)$ such that

$$x_k \to L \ (k \to \infty),$$

in the sense that

$$g(x_k - L) \to 0 \ (k \to \infty),$$

Then, since the inequality

$$g(x_k) \leq g(x_k - L) + g(L)$$

holds by subadditivity of g, the sequence $\{g(x_k)\}$ is bounded.
Therefore,

$$\begin{aligned} g\big[(\lambda_k x_k - \lambda L)\big] &= g\big[(\lambda_k x_k - \lambda x_k + \lambda x_k - \lambda L)\big] \\ &= g\big[(\lambda_k - \lambda)x_k + \lambda(x_k - L)\big] \\ &\leq g\big[(\lambda_k - \lambda)x_k\big] + g\big[\lambda(x_k - L)\big] \\ &\leq \mid (\lambda_k - \lambda) \mid g(x_k) + \mid \lambda \mid g(x_k - L) \to 0 \end{aligned}$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous.
Hence $\mathcal{M}_S^I(f)$ is a paranormed space.
For $\mathcal{M}_{S_0}^I(f)$, the result is similar.

Theorem 2.2.3. A sequence $x = (x_k) \in \ell_\infty$ I-converges if and only if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\left\{k \in \mathbb{N} : f\Big(\mid T(x_k) - T(x_{N_\epsilon}) \mid \Big) < \epsilon \right\} \in \mathcal{L}(I). \tag{2.2.10}$$

Proof. Let $x = (x_k) \in \ell_\infty$.
Suppose that $L = I - \lim x$. Then, the set

$$B_\epsilon = \left\{k \in \mathbb{N} : f\Big(\mid T(x_k) - L \mid \Big) < \frac{\epsilon}{2} \right\} \in \mathcal{L}(I) \ \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have,

$$f\Big(\mid T(x_k) - T(x_{N_\epsilon}) \mid \Big) \leq f\Big(\mid T(x_k) - L \mid \Big) + f\Big(\mid T(x_{N_\epsilon}) - L \mid \Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.
Hence $\left\{k \in \mathbb{N} : f\Big(\mid T(x_k) - T(x_{N_\epsilon}) \mid \Big) < \epsilon\right\} \in \pounds(I)$
Conversely, suppose that

$$\left\{k \in \mathbb{N} : f\Big(\mid T(x_k) - T(x_{N_\epsilon}) \mid \Big) < \epsilon\right\} \in \pounds(I).$$

That is $\left\{k \in \mathbb{N} :\mid f\big(|T(x_k)|\big) - f\big(|T(x_{N_\epsilon})\mid\big) \mid< \epsilon\right\} \in \pounds(I)$, for all $\epsilon > 0$. Then, the set

$$C_\epsilon = \left\{k \in \mathbb{N} : f\big(|T(x_k)|\big) \in [f\big(|T(x_{N_\epsilon})\mid\big) - \epsilon, f\big(|T(x_{N_\epsilon})\mid\big) + \epsilon]\right\} \in \pounds(I) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = \Big[f\big(|T(x_{N_\epsilon})\big) - \epsilon, f\big(|T(x_{N_\epsilon})\big) + \epsilon\Big]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in \pounds(I)$ as well as $C_{\frac{\epsilon}{2}} \in \pounds(I)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \pounds(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi.$$

That is

$$\{k \in \mathbb{N} : f\big(|T(x_k)|\big) \in J\} \in \pounds(I).$$

That is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \ldots\ldots \supseteq I_k \supseteq \ldots\ldots\ldots\ldots$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : f\big(|T(x_k)|\big) \in I_k\} \in \pounds(I)$ for (k=1,2,3,4,......).
Then, there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim f\big(|T(x_k)|\big)$ showing that $x = (x_k) \in \ell_\infty$ is I-convergent.
Hence the result.

Theorem 2.2.4. Let f_1 and f_2 be two modulus functions and satisfying $\Delta_2 -$ Condition, then

(a) $\mathcal{X}(f_2) \subseteq \mathcal{X}(f_1 f_2)$,
(b) $\mathcal{X}(f_1) \cap (f_2) \subseteq \mathcal{X}(f_1 + f_2)$
for $\mathcal{X}= \mathcal{S}^I$, $\mathcal{S}^I_\circ$, $\mathcal{M}^I_{\mathcal{S}}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$

Proof.(a) Let $x = (x_k) \in \mathcal{S}^I_\circ(f_2)$ be any arbitrary element. Then, the set

$$\left\{k \in \mathbb{N} : f_2\Big(\mid T(x_k) \mid \Big) \geq \epsilon\right\} \in I. \tag{2.2.11}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_1(t) < \epsilon$, $0 \leq t \leq \delta$.
Let us denote

$$y_k = f_2\Big(\mid T(x_k) \mid \Big)$$

and consider

$$\lim_k f_1\big(y_k\big) = \lim_{y_k \leq \delta, k \in \mathbb{N}} f_1\big(y_k\big) + \lim_{y_k > \delta, k \in \mathbb{N}} f_1\big(y_k\big).$$

Now, since f_1 is an modulus function , we have

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} f_1\big(y_k\big) \leq f_1\big(2\big) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k). \tag{2.2.12}$$

For $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Now, since f_1 is non-decreasing and modulus, it follows that

$$f_1(y_k) < f_1(1 + \frac{y_k}{\delta}) < \frac{1}{2} f_1(2) + \frac{1}{2} f_1(\frac{2y_k}{\delta}).$$

Again, since f_1 satisfies $\Delta_2 -$ Condition, we have

$$f_1\big(y_k\big) < \frac{1}{2} K \frac{(y_k)}{\delta} f_1\big(2\big) + \frac{1}{2} K \frac{(y_k)}{\delta} f_1(2).$$

Thus, $f_1\big(y_k\big) < K \frac{(y_k)}{\delta} f_1\big(2\big)$
Hence,

$$\lim_{y_k > \delta, k \in \mathbb{N}} f_1\big(y_k\big) \leq \max\{1, K\delta^{-1} f_1\big(2\big) \lim_{y_k > \delta, k \in \mathbb{N}} (y_k). \tag{2.2.13}$$

Therefore, from (2.2.11), (2.2.12) and (2.2.13), we have
$(x_k) \in \mathcal{S}^I_\circ(f_1 f_2)$
Thus, $\mathcal{S}^I_\circ(f_2) \subseteq \mathcal{S}^I_\circ(f_1 f_2)$. Hence, $\mathcal{X}(f_2) \subseteq \mathcal{X}(f_1 f_2)$ for $\mathcal{X}= \mathcal{S}^I_\circ$.
For $\mathcal{X}= \mathcal{S}^I$, $\mathcal{M}^I_\mathcal{S}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$ the inclusions can be established similarly.
(b). Let $x = (x_k) \in \mathcal{S}^I_\circ(f_1) \cap \mathcal{S}^I_\circ(f_2)$. Let $\epsilon > 0$ be given. Then, the sets

$$\left\{k \in \mathbb{N} : f_1\Big(\mid T(x_k) \mid \Big) \geq \epsilon\right\} \in I \tag{2.2.14}$$

and

$$\left\{k \in \mathbb{N} : f_2\Big(\mid T(x_k) \mid \Big) \geq \epsilon\right\} \in I. \tag{2.2.15}$$

Therefore, from (2.2.14) and (2.2.15) the set

$$\left\{k \in \mathbb{N} : (f_1 + f_2)\Big(\mid T(x_k) \mid \Big) \geq \epsilon\right\} \in I.$$

Thus, $x = (x_k) \in \mathcal{S}^I_\circ(f_1 + f_2)$.
Hence, $\mathcal{S}^I_\circ(f_1) \cap \mathcal{S}^I_\circ(f_2) \subseteq \mathcal{S}^I_\circ(f_1 + f_2)$
For $\mathcal{X}= \mathcal{S}^I$, $\mathcal{M}^I_\mathcal{S}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$ the inclusions are similar.

For $f_2(x) = x$ and $f_1(x) = f(x)$, $\forall\, x \in [0,\infty)$, we have the following corollary.

Corollary 2.2.5. $\mathcal{X} \subseteq \mathcal{X}(f)$ for $\mathcal{X}= \mathcal{S}^I$, $\mathcal{S}^I_\circ$, $\mathcal{M}^I_\mathcal{S}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$

Theorem 2.2.6. For any modulus function f, the spaces $\mathcal{S}^I_\circ(f)$ and $\mathcal{M}^I_{\mathcal{S}_\circ}(f)$ are solid and monotone.

Proof. we prove the result for the space $\mathcal{S}^I_\circ(f)$. For $\mathcal{M}^I_{\mathcal{S}_\circ}(f)$, the proof can be obtained similarly.
For, let $(x_k) \in \mathcal{S}^I_\circ(f)$ be any arbitrary element. Then, the set

$$\{k \in \mathbb{N} : f\Big(\mid T(x_k) \mid \Big) \geq \epsilon\} \in I. \tag{2.2.16}$$

Let (α_k) be a sequence of scalars such that

$$\mid \alpha_k \mid \leq 1, \text{for all k} \in \mathbb{N}.$$

Then the result follows from (2.2.16) and the following inequality.

$$f\Big(\mid T(\alpha_k x_k)\mid\Big) = f\Big(\mid \alpha_k T(x_k)\mid\Big) \leq \mid \alpha_k \mid f\Big(\mid T(x_k)\mid\Big) \leq f\Big(\mid T(x_k)\mid\Big), \text{ for all k} \in \mathbb{N}.$$

That the space $\mathcal{S}^I_\circ(f)$ is monotone follows from the Lemma (I). Hence $\mathcal{S}^I_\circ(f)$ is solid and monotone.

Theorem 2.2.7. The spaces $\mathcal{S}^I(f)$ and $\mathcal{M}^I_S(f)$ are not neither solid nor monotone.

Proof. Here we give a counter example for the proof of this result.

Counter example. Let $I = I_f$ and $f(x) = x$ for all $x \in [0, \infty)$. Consider the K-step $\mathcal{Z}_K$ of $\mathcal{Z}$ defined as follows.
Let $(x_k) \in \mathcal{Z}$ and let $(y_k) \in \mathcal{Z}_K$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined as by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{S}^I(f)$ and $\mathcal{M}^I_S(f)$ but its K-step preimage does not belong to $\mathcal{S}^I(f)$ and $\mathcal{M}^I_S(f)$.
Thus, $\mathcal{S}^I(f)$ and $\mathcal{M}^I_S(f)$ are not monotone. Hence, $\mathcal{S}^I(f)$ and $\mathcal{M}^I_S(f)$ are not solid by lemma(I).

Theorem 2.2.8. If $(x = x_k)$ and $(y = y_k)$ be two sequences with $T(x.y) = T(x)T(y)$. Then, the spaces $\mathcal{S}^I(f)$ and $\mathcal{S}^I_\circ(f)$ are sequence algebra.

Proof. Let $(x = x_k)$ and $(y = y_k)$ be two elements of $\mathcal{S}^I_\circ(f)$ with $T(x.y) = T(x)T(y)$. Then, the sets

$$\Big\{k \in \mathbb{N} : f\Big(\mid T(x_k)\mid\Big) \geq \epsilon\Big\} \in I \tag{2.2.17}$$

and

$$\Big\{k \in \mathbb{N} : f\Big(\mid T(y_k)\mid\Big) \geq \epsilon\Big\} \in I. \tag{2.2.18}$$

Therefore,

$$\Big\{k \in \mathbb{N} : f\Big(\mid T(x_k).T(y_k)\mid\Big) \geq \epsilon\Big\} \in I.$$

Thus, $(x_k).(y_k) \in \mathcal{S}_o^I(f)$.
Hence, $\mathcal{S}_o^I(f)$is sequence algebra. For $\mathcal{S}^I(f)$, the result can be proved similarly.

Theorem 2.2.9. Let f be a modulus function. Then, $\mathcal{S}_o^I(f) \subset \mathcal{S}^I(f) \subset \mathcal{S}_\infty^I(f)$.

Proof. The inclusion $\mathcal{S}_o^I(f) \subset \mathcal{S}^I(f)$ is obvious.
Next, let $(x_k) \in \mathcal{S}^I(f)$. Then there exists some L such that

$$\{k \in \mathbb{N} : f(\mid T(x_k) - L \mid) \geq \epsilon\} \in I.$$

We have

$$f(\mid T(x_k) \mid) \leq \frac{1}{2} f(\mid T(x_k) - L \mid) + f(\frac{1}{2} \mid L \mid)$$

Taking supremum over k on both sides, we get $(x_k) \in \mathcal{S}_\infty^I(f)$
Hence, $\mathcal{S}_o^I(f) \subset \mathcal{S}^I(f) \subset \mathcal{S}_\infty^I(f)$

Theorem 2.2.10. If $f(x) = x$ for all $x \in [0, \infty]$. Then, the function $\hbar : \mathcal{M}_S^I(f) \rightarrow \mathbb{R}$ defined by $\hbar(x) = I - \lim f(\mid T(x_k) \mid)$, where $\mathcal{M}_S^I(f) = \mathcal{S}_\infty(f) \cap \mathcal{S}^I(f)$ is a Lipschitz function and hence uniformly continuous.

Proof. Clearly the function $\hbar$ is well defined. Let $x = (x_k),\ y = (y_k) \in \mathcal{M}_S^I(f),\ x \neq y$. Then, the sets

$$A_x = \{k \in \mathbb{N} : f(\mid T(x) - \hbar(x) \mid) \geq \| x - y \|_* \} \in I.$$

$$A_y = \{k \in \mathbb{N} : f(\mid T(y) - \hbar(y) \mid) \geq \| x - y \|_* \} \in I.$$

where

$$\| x - y \|_* = \sup_k f(\mid T(x_k) - T(y_k) \mid)$$

Thus, the sets

$$B_x = \{k \in \mathbb{N} : \mid T(x) - \hbar(x) \mid < \| x - y \|_* \} \in \mathcal{L}(I).$$

$$B_y = \{k \in \mathbb{N} : \mid T(y) - \hbar(y) \mid < \| x - y \|_* \} \in \mathcal{L}(I).$$

Hence, $B = B_x \cap B_y \in \mathcal{L}(I)$, so that $B \neq \emptyset$
Now, taking $k \in B$, we have

$$\mid \hbar(x) - \hbar(y) \mid \leq \mid \hbar(x) - T(x_k) \mid + \mid T(x_k) - T(y_k) \mid + \mid T(y_k) - \hbar(y) \mid \leq 3 \| x - y \|_* .$$

Therefore, $\hbar$ is Lipschitz function and hence uniformly continuous.

Theorem 2.2.11. If $f(x) = x$ for all $x \in [0, \infty]$ and if $x = (x_k)$, $y = (y_k) \in \mathcal{M}_S^I(f)$ with $T(x.y) = T(x)T(y)$. Then $(x.y) \in \mathcal{M}_S^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$ where $\hbar : \mathcal{M}_S^I(f) \to \mathbb{R}$ is defined by $\hbar(x) = I - \lim f(| T(x_k) |)$.

Proof. For $\epsilon > 0$, the sets

$$B_x = \{k \in \mathbb{N} : |T(x_k) - \hbar(x)| < \epsilon\} \in \mathcal{L}(I), \tag{2.2.19}$$

$$B_y = \{k \in \mathbb{N} : |T(y_k) - \hbar(y)| < \epsilon\} \in \mathcal{L}(I). \tag{2.2.20}$$

where $\| x - y \|_* = \epsilon$
Now,

$$|T(x_k y_k) - \hbar(x)\hbar(y)| = |T(x_k)T(y_k) - T(x_k)\hbar(y) + T(x_k)\hbar(y) - \hbar(x)\hbar(y)|$$

$$\leq |T(x_k)||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \tag{2.2.21}$$

As $\mathcal{M}_S^I(f) \subseteq \mathcal{S}_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|T(x_k)| < M$ and $|\hbar(y)| < M$. Therefore, from (2.2.19), (2.2.20) and(2.2.21), we have

$$|T(x_k y_k) - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

for all $k \in B_x \cap B_y \in \mathcal{L}(I)$.
Hence $(x.y) \in \mathcal{M}_S^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

CHAPTER 3

ON BV_σ I-CONVERGENT SEQUENCE SPACES DEFINED BY AN ORLICZ FUNCTION

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"Aurel Vlaicu" University of Arad
Faculty of Exact Sciences
Romania

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CHAPTER-3

ON BV_σ I-CONVERGENT SEQUENCE SPACES DEFINED BY AN ORLICZ FUNCTION

3.1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively with norm

$$\|x\| = \sup_k \mid x_k \mid .$$

Let v be denote the space of sequences of bounded variation. That is,

$$v = \left\{ x = (x_k) : \sum_{k=o}^{\infty} \mid x_k - x_{k-1} \mid < \infty, x_{-1} = 0 \right\}. \tag{3.1.1}$$

v is a Banach Space normed by

$$\|x\| = \sum_{k=0}^{\infty} \mid x_k - x_{k-1} \mid . \quad (see[76])$$

Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if

(i) $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k.
(ii) $\phi(e) = 1$ where $e = \{1, 1, 1, ...\}$,
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

''A mathematician is a person who can find analogies between theorems, a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories."-*Stefan Banach*

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_k) : \lim_{m \to \infty} t_{m,k}(x) = L \text{ uniformly in k}, L = \sigma - \lim x \right\} \tag{3.1.2}$$

where $m \geq 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} \dots + x_{\sigma^m(k)}}{m+1} \text{ and } t_{-1,k} = 0, \tag{3.1.3}$$

where $\sigma_m(k)$ denote the m^{th}-iterate of $\sigma(k)$ at k. In case σ is the translation mapping, that is, $\sigma(k) = k + 1$, σ-mean is called a Banach limit(see,[4]) and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences.The special case of (3.1.2) in which $\sigma(k) = k+1$ was given by Lorentz (see[69], Theorem 1), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c (see[69]) in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{3.1.4}$$

Remark 3.1.1. In view of above discussion we have $c \subset V_\sigma$.

Theorem 3.1.2. (see[76],Theorem 1.1). A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits. That is, if and only if for all $k \geq 0, j \geq 1$, $\sigma^j(k) \neq k$.

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{3.1.5}$$

assuming that $t_{-1,k}(x) = 0$.

A straight forward calculation shows that

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if}(m \geq 1), \\ x_k & \text{if}(m = 0) \end{cases} \tag{3.1.6}$$

For any sequence x, y and scalar λ, we have

$$\phi_{m,k}(x+y) = \phi_{m,k}(x) + \phi_{m,k}(y)$$

and

$$\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x).$$

Definition 3.1.3. A sequence $x \in \ell_\infty$ is of σ-bounded variation if and only if
(i) $\sum_{m=0}^{\infty} \mid \phi_{m,k}(x) \mid$ converges uniformly in k.
(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

Subsequently invariant means have been studied by Mursaleen[76],[77], Ahmad and Mursaleen[1], Raimi [89], Khan and Ebadullah [49],[43], Schafer[97] and many others. Mursaleen [77] defined the sequence space BV_σ, the space of all sequences of σ -bounded variation as

$$BV_\sigma = \{x \in \ell_\infty : \sum_m \mid \phi_{m,k}(x) \mid < \infty, \text{uniformly in k}\}.$$

Theorem 3.1.4. BV_σ is a Banach space normed by

$$\| x \| = \sup_k \sum \mid \phi_{m,k}(x) \mid (see[77]).$$

Definition 3.1.5.(see[101]) A function $M : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it satisfies the following conditions;
(i) M is continuous, convex and non-decreasing,
(ii) $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$

Remark 3.1.6.(see[101]) If the convexity of an Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark 3.1.7. If M is an Orlicz function,then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy Δ_2-Condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leq \text{KL}M(u)$ for all values of $L > 1$(see[101]).

Lindenstrauss and Tzafriri[68] used the idea of an Orlicz function to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}. \tag{3.1.7}$$

The space ℓ_M becomes a Banach space with the norm

$$\| x \| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\} \tag{3.1.8}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = t^P$ for $1 \leq p < \infty$.

Later on, some Orlicz sequence spaces were investigated by Parashar and Choudhury[87], Maddox[71], Hazarika, *et,al* [26], Bhardwaj and Singh[6], Tripathy and Hazarika[101] and many others.

Initially, as a generalization of statistical convergence(see[16],[18]), the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyńki ([63],[65]). Later on, it was studied by Khan and Ebadullah[49],[43], Hazarika and Esi[25], Šalát, Tripathy and Ziman[95],[96], Demirci [11] and many others.

Here we give some preliminaries about the notion of I-convergence.

Definition 3.1.8. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : |x_k - L| \geq \epsilon, n \leq k\}| = 0$$

where vertical lines denote the cardinality of the enclosed set.

Definition 3.1.9. Let $\mathbb{N}$ be the set of natural numbers. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if

1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$,

2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 3.1.10. A non-empty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if

1) $\Phi \notin \pounds(I)$,

2) $\forall\ A,\ B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$,

3) $\forall\ A \in \pounds$(I) and $A \subseteq B \Rightarrow B \in \pounds(I)$.

Definition 3.1.11. An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial if $I \neq 2^{\mathbb{N}}$.

Definition 3.1.12. A non-trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$.

Definition 3.1.13. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 3.1.14. For each ideal I, there is a filter $\pounds(I)$ corresponding to I.
i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 3.1.15. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 3.1.16. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 3.1.17. A sequence $x = (x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 3.1.18. A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 3.1.19. A sequence space E is said to be solid(normal) if $(\alpha_k x_k) \in E$

whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$.

Definition 3.1.20. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$

Definition 3.1.21. A sequence space is$E$ said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 3.1.22. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 3.1.23. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 3.1.24. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 3.1.25. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 3.1.26. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then, I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 3.1.27. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 3.1.28. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 3.1.29. If $I_\delta - \lim x_k = l$, then $I_d - \lim x_k = l$

The following lemmas remained an important tool for the establishment of some results of this article.

Lemma(I)(see[101]). Every solid space is monotone

Lemma(II)(see[101]). Let $K \in \mathcal{L}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

Lemma(III)(see][101]). If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

3.2. MAIN RESULTS

Recently, Khan and K.Ebadullah[18] introduced and studied the following sequence space.
For $m \geq 0$

$$BV_\sigma^I = \left\{ x = (x_k) \in \omega : \{k \in \mathbb{N} :\mid \phi_{m.k}(x) - L \mid \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C} \right\} \tag{3.2.1}$$

In this article we introduce the following sequence spaces.
For $m \geq 0$

$$BV_\sigma^I(M) = \left\{ x = (x_k) \in \omega : I\text{--}\lim M(\frac{\mid \phi_{m,k}(x) - L \mid}{\rho}) = 0, \text{for some L} \in \mathbb{C}, \rho > 0 \right\}, \tag{3.2.2}$$

$$_0BV_\sigma^I(M) = \left\{ x = (x_k) \in \omega : I - \lim M(\frac{\mid \phi_{m,k}(x) \mid}{\rho}) = 0, \rho > 0 \right\}, \tag{3.2.3}$$

$$_\infty BV_\sigma^I(M) = \left\{ x = (x_k) \in \omega : \{k \in \mathbb{N} : \exists K > 0 s.t. M(\frac{\mid \phi_{m,k}(x) \mid}{\rho}) \geq K\} \in I, \rho > 0 \right\}, \tag{3.2.4}$$

$$_\infty BV_\sigma(M) = \left\{ x = (x_k) \in \omega : \sup M(\frac{\mid \phi_{m,k}(x) \mid}{\rho}) < \infty, \rho > 0 \right\}. \tag{3.2.5}$$

We also denote
$\mathcal{M}^I_{BV_\sigma}(M) = BV^I_\sigma(M) \cap {}_\infty BV_\sigma(M)$.
and
${}_0\mathcal{M}^I_{BV_\sigma}(M) = {}_0BV^I_\sigma(M) \cap {}_\infty BV_\sigma(M)$.

Throughout the article, if required, we denote
$\phi_{m,k}(x)=x'$, $\phi_{m,k}(y)=y'$ and $\phi_{m,k}(z)=z'$ where x, y, z are (x_k), (y_k) and (z_k) respectively.

Theorem 3.2.1. For any Orlicz function M, the classes of sequence ${}_0BV^I_\sigma(M)$, $BV^I_\sigma(M)$; ${}_0\mathcal{M}^I_{BV_\sigma}(M)$ and $\mathcal{M}^I_{BV_\sigma}(M)$ are the linear spaces.

Proof. We shall prove the result for the space $BV^I_\sigma(M)$, others will follow similarly. For, let $x = (x_k), y = (y_k) \in BV^I_\sigma(M)$ be any two arbitrary elements and let α, β are scalars.
Now, since $(x_k), (y_k) \in BV^I_\sigma(M) \Rightarrow \exists$ some +ve numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1,\ \rho_2 > 0$ such that

$$I - \lim_k M\left(\frac{\mid \phi_{m,k}(x) - L_1 \mid}{\rho_1}\right) = 0 \tag{3.2.6}$$

and

$$I - \lim_k M\left(\frac{\mid \phi_{m,k}(y) - L_2 \mid}{\rho_2}\right) = 0 \tag{3.2.7}$$

$\Rightarrow$ for any given $\epsilon > 0$, the sets

$$A_1 = \left\{k \in \mathbb{N} : M\left(\frac{\mid \phi_{m,k}(x) - L_1 \mid}{\rho_1}\right) > \frac{\epsilon}{2}\right\} \in I \tag{3.2.8}$$

and

$$A_2 = \left\{k \in \mathbb{N} : M\left(\frac{\mid \phi_{m,k}(y) - L_2 \mid}{\rho_2}\right) > \frac{\epsilon}{2}\right\} \in I. \tag{3.2.9}$$

Let $\rho_3 = \max\{2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2\}$.

(3.2.10)

Since, M is non-decreasing and convex function, we have

$$M\left(\frac{\mid (\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2) \mid}{\rho_3}\right)$$

$$\leq M\left(\frac{|\alpha||x'_k - L_1|}{\rho_3}) + M(\frac{|\beta||y'_k - L_2|}{\rho_3}\right)$$

$$\leq M\left(\frac{|x'_k - L_1|}{\rho_1}) + M(\frac{|y'_k - L_2|}{\rho_2}\right). \tag{3.2.11}$$

Therefore, from (3.2.8), (3.2.9) and (3.2.11), we have

$$\left\{k \in \mathbb{N} : M\left(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) > \epsilon\right\} \subseteq A_1 \cup A_2 \in I.$$

implies that

$$\left\{k \in \mathbb{N} : M\left(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) > \epsilon\right\} \in I.$$

That is,

$$I - \lim M\left(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) = 0.$$

Thus, $\alpha x_k + \beta y_k \in BV_\sigma^I(M)$. But $(x_k), (y_k) \in BV_\sigma^I(M)$ are the arbitrary elements Therefore, $\alpha x_k + \beta y_k \in BV_\sigma^I(M)$, for all $(x_k), (y_k) \in BV_\sigma^I(M)$ and for all scalars α, β.

Hence, $BV_\sigma^I(M)$ is linear.

Theorem 3.2.2. Let M_1 and M_2 be two Orlicz functions and satisfying Δ_2- Condition, then

(a) $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1 M_2)$,

(b) $\mathcal{X}(M_1) \cap (M_2) \subseteq \mathcal{X}(M_1 + M_2)$

for $\mathcal{X}$= ${}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.

Proof.(a) Let $x = (x_k) \in {}_0BV_\sigma^I(M_2)$ be any arbitrary element$\Rightarrow \exists\ \rho > 0$ such that

$$I - \lim M_2\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = 0. \tag{3.2.12}$$

i.e.

$$I-\lim M_2\left(\frac{|x'_k|}{\rho}\right)=0.$$

Let $\epsilon>0$ and choose δ with $0<\delta<1$ such that $M_1(t)<\epsilon,\ 0\le t\le\delta$.
Let us write $y_k=M_2(\frac{|x'_k|}{\rho})$ and consider

$$\lim_k M_1(y_k)=\lim_{y_k\le\delta,k\in\mathbb{N}} M_1(y_k)+\lim_{y_k>\delta,k\in\mathbb{N}} M_1(y_k).$$

Now, since M_1 is an Orlicz function ,we have
$M_1(\lambda x)\le\lambda M_1(x)$ for all λ with $0<\lambda<1$.
Therefore,

$$\lim_{y_k\le\delta,k\in\mathbb{N}} M_1(y_k)\le M_1(2)\lim_{y_k\le\delta,k\in\mathbb{N}}(y_k). \tag{3.2.13}$$

For $y_k>\delta$, we have $y_k<\frac{y_k}{\delta}<1+\frac{y_k}{\delta}$.
Now, since M_1 is non-decreasing and convex, it follows that

$$M_1(y_k)<M_1(1+\frac{y_k}{\delta})<\frac{1}{2}M_1(2)+\frac{1}{2}M_1(\frac{2y_k}{\delta}).$$

Again, since M_1 satisfies Δ_2- Condition, we have

$$M_1(y_k)<\frac{1}{2}K\frac{(y_k)}{\delta}M_1(2)+\frac{1}{2}K\frac{(y_k)}{\delta}M_1(2).$$

Thus, $M_1(y_k)<K\frac{(y_k)}{\delta}M_1(2)$.
Hence,

$$\lim_{y_k>\delta,k\in\mathbb{N}} M_1(y_k)\le\max\{1,K\delta^{-1}M_1(2)\lim_{y_k>\delta,k\in\mathbb{N}}(y_k). \tag{3.2.14}$$

Therefore, from (3.2.12), (3.2.13) and (3.2.14), we have

$$I-\lim_k M_1(y_k)=0$$

i.e.

$$I-\lim_k M_1M_2(\frac{|\phi_{m,k}(x)|}{\rho})=0$$

implies that
$(x_k)\in {}_0BV_\sigma^I(M_1M_2)$.

Thus, ${}_0BV_\sigma^I(M_2) \subseteq {}_0BV_\sigma^I(M_1M_2)$. Hence, $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1M_2)$ for $\mathcal{X}= {}_0BV_\sigma^I$. For $\mathcal{X} = BV_\sigma^I$,$\mathcal{X}= {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X} = \mathcal{M}_{BV_\sigma^I}$ the inclusions can be established similarly.

(b). Let $x = (x_k) \in {}_0BV_\sigma^I(M_1) \cap {}_0BV_\sigma^I(M_2)$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the sets

$$I - \lim M_1\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right) = 0$$

and

$$I - \lim M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right) = 0.$$

Therefore,

$$I - \lim M_1 + M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right) = I - \lim M_1\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right) + I - \lim M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right)$$

implies that

$$I - \lim M_1 + M_2\left(\frac{\mid \phi_{m,k}(x) \mid}{\rho}\right) = 0.$$

Thus, $x = (x_k) \in {}_0BV_\sigma^I(M_1 + M_2)$
Hence, ${}_0BV_\sigma^I(M_1) \cap {}_0BV_\sigma^I(M_2) \subseteq {}_0BV_\sigma^I(M_1 + M_2)$.
For $\mathcal{X} = BV_\sigma^I$, $\mathcal{X}= {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X} = \mathcal{M}_{BV_\sigma}^I$ the inclusions are similar.

For $M_2(x) = (x)$ and $M_1(x) = M(x)$, $\forall\ x \in [0, \infty)$, we have the following corollary.

Corollary.$\mathcal{X} \subseteq \mathcal{X}(M)$ for $\mathcal{X}= {}_0BV_\sigma^I$; BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.

Theorem 3.2.3. For any orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and ${}_0\mathcal{M}_{BV_\sigma}^I$ are solid and monotone.

Proof. Here we consider ${}_0BV_\sigma^I(M)$ and for ${}_0\mathcal{M}_{BV_\sigma}^I$ the proof shall be similar.
For, let $(x_k) \in_0 BV_\sigma^I(M)$ be any arbitrary element. $\Rightarrow$ $\exists\rho > 0$ such that
$I - \lim\limits_k M\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = 0.$

Let (α_k) be a sequence of scalars such that $| \alpha_k | \leq 1$, for all k $\in \mathbb{N}$.
Now, since M is an Orlicz function.
Therefore,

$$M\left(\frac{| \alpha_k \phi_{m,k}(x) |}{\rho}\right) \leq | \alpha_k | M\left(\frac{| \phi_{m,k}(x) |}{\rho}\right) \leq M\left(\frac{| \phi_{m,k}(x) |}{\rho}\right)$$

$$\Rightarrow M\left(\frac{| \alpha_k \phi_{m,k}(x) |}{\rho}\right) \leq M\left(\frac{| \phi_{m.k}(x) |}{\rho}\right), \text{for all k} \in \mathbb{N}$$

implies that $I - \lim_k M\left(\frac{|\alpha_k \phi_{m,k}(x)|}{\rho}\right) = 0$.
Thus, $(\alpha_k x_k) \in_0 BV_\sigma^I(M)$.
Hence ${}_0BV_\sigma^I(M)$ is solid.
Therefore, by lemma(I), ${}_0BV_\sigma^I(M)$ is monotone. Hence the result.

Theorem 3.2.4. For any orlicz function M, the spaces $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ are neither solid nor monotone in general.

Proof. Here we give counter example for the establishment of this result.
For, let us consider $I = I_f$ and $M(x) = x$, for all $x \in [0, \infty)$.
Consider, the K-step space $B_K(M)$ of $B(M)$ as follows.
Let$(x_k) \in B(M)$ and $(y_k) \in B_K(M)$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined as $x_k = 1$, for all k $\in \mathbb{N}$, then $x_k \in BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ but its K-step space pre-image does not belong to $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$. Thus, $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ are not monotone and hence by lemma(I) they are not solid.

Theorem 3.2.5. For an Orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are not convergence free.

Proof Let $I = I_f$ and $M(x) = x$ for all x $\in [0, \infty)$. Consider the sequences (x_k) and (y_k) defined as follows.

$$x_k = \frac{1}{k} \text{ and } y_k = k, \text{ for all k} \in \mathbb{N}.$$

Then, (x_k) belongs to both $\in {}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ but (y_k) does not belongs to both ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$.
Hence, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are not convergence free.

Theorem 3.2.6. For an Orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are sequence algebra.

Proof. Here we consider ${}_0BV_\sigma^I(M)$. For the other one, result is similar.
Let $x = (x_k), y = (y_k) \in {}_0BV_\sigma^I(M)$ be any two arbitrary elements.
$\Rightarrow \exists\ \rho_1, \rho_2 > 0$ such that

$$I - \lim_k M\Big(\frac{|\ \phi_{m,k}(x)\ |}{\rho_1}\Big) = 0$$

and

$$I - \lim_k M\Big(\frac{|\ \phi_{m,k}(y)\ |}{\rho_2}\Big) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$.
Then, it is obvious that

$$I - \lim_k M\Big(\frac{|\ \phi_{m,k}(x)\phi_{m,k}(y)\ |}{\rho}\Big) = 0$$

implies that

$$(x_k.y_k) = (x_k y_k) \in {}_0BV_\sigma^I(M).$$

Hence, ${}_0BV_\sigma^I(M)$ is a Sequence algebra.

Theorem 3.2.7. Let M be an Orlicz function. Then,
${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M) \subseteq {}_\infty BV_\sigma^I(M)$.

Proof. Let M be an Orlicz function. Then, we have to show that
${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M) \subseteq {}_\infty BV_\sigma^I(M)$.
Firstly, ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M)$ is obvious.
Now, let $x = (x_k) \in BV_\sigma^I(M)$ be any arbitrary element.
$\Rightarrow \exists\ \rho > 0$ such that $I - \lim_k M\big(\frac{|\phi_{m,k}(x) - L|}{\rho}\big) = 0$ for some $L \in \mathbb{N}$.

Now,
$M\left(\frac{|\phi_{m,k}(x)|}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{|\phi_{m,k}(x)-L|}{\rho}\right) + \frac{1}{2}M\left(\frac{|L|}{\rho}\right)$
Taking supremum over k to both sides, we have
$x = (x_k) \in {}_{\infty}BV_{\sigma}^{I}(M)$.
Hence, ${}_{0}BV_{\sigma}^{I}(M) \subseteq BV_{\sigma}^{I}(M) \subseteq {}_{\infty}BV_{\sigma}^{I}(M)$.

CHAPTER 4

ON PARANORM $\mathcal{I}$-CONVERGENT SEQUENCE SPACES OF BOUNDED OPERATORS DEFINED BY A MODULUS FUNCTION

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Emails: rsaadati@aut.ac.ir
rsaadati@eml.cc

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CHAPTER-4

ON PARANORM $\mathcal{I}$-CONVERGENT SEQUENCE SPACES OF BOUNDED OPERATORS DEFINED BY A MODULUS FUNCTION

4.1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers, respectively.

We denote the space of all real or complex sequences by

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

Any subspace λ of the linear space ω of sequences is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous, for all $i \in \mathbb{N}$. A space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space.

We denote the space of all linear operators from a normed space X to normed space Y by

$$\mathcal{L}(\mathcal{T}) = \left\{\mathcal{T} = (T_k) : T_k : X \to Y \text{ is linear, for each k} \in \mathbb{N}\right\}. \tag{4.1.1}$$

Definition 4.1.1. Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator T is said to be bounded, if there exists a real $k > 0$ such that

$$\| Tx \| \leq k \| x \|, \text{ for all, } x \in \mathcal{D}(T). \tag{4.1.2}$$

The sequences of bounded linear operators arise frequently in the abstract formulation of concrete situations. For instance, in connection with convergence problems

''Mathematics is a free flow of thoughts and concepts which a mathematician, in the same way as musician does with the tones of music and a poet with words, puts together into theorems and theories'' -*Orlicz*.

of Fourier series or sequences of interpolation polynomials or methods of numerical integration, to name just a few. In such cases one is usually concerned with the convergence of those sequences of operators with boundedness of corresponding sequences of norms or with similar properties. We also study $\mathcal{I}$-convergence of sequences of these operators and some related results.

Let $\mathcal{B}_{\infty}(\mathcal{T})$ be denote the normed space of sequences of all bounded linear operators from a normed space X to a normed space Y, normed by

$$\| \mathcal{T} \| = \sup_k \| T_k(x) \| \quad (\text{see}[67]). \tag{4.1.3}$$

$\mathcal{B}_{\infty}(\mathcal{T})$ is a Banach space if Y is a Banach space.

Throughout, O and I represent zero and identity operators, respectively.

It was due to J.Von Neumann (see [84]), the following definitions were introduced.

Definition 4.1.2. Let X and Y be two normed linear spaces. A sequence (T_k) of operators $T_k \in \mathcal{B}_{\infty}(\mathcal{T})$ is said to be;
1) Uniformly convergent, if (T_k) converges in the norm on $\mathcal{B}_{\infty}(\mathcal{T})$.
2) Strongly convergent, if $(T_k x)$ converges strongly in Y for every $x \in X$.
3) Weakly convergent, if $(T_k x)$ converges weakly in Y for every $x \in X$.

That is, if there is a linear operator $T : X \to Y$ such that $T_k \in \mathcal{B}_{\infty}(\mathcal{T})$ is;
1) Uniformly convergent, if $\| T_k - T \| \to 0$.
2) Strongly convergent, if $\| T_k x - Tx \| \to 0$, for every $x \in X$.
3) Weakly convergent, if $| f(T_k x) - f(Tx) | \to 0$, for every $x \in X$ and $f \in Y'$.

Theorem 4.1.3. Let $T_k \in \mathcal{B}_{\infty}(\mathcal{T})$, where k=1,2,3,.... Then, $T_k \to T$, if and only if, for every $\epsilon > 0$, there is an N depending only on ϵ, such that for all $k > N$ and all $x \in X$ of norm 1, we have,

$$\| T_k x - Tx \| < \epsilon (\text{ see}[67]). \tag{4.1.4}$$

Let $\mathcal{C}(\mathcal{T})$ and $\mathcal{C}_0(\mathcal{T})$ be the convergent and null sequence spaces, respectively of the sequence (T_k) of bounded operators defined as follows.

$$\mathcal{C}(\mathcal{T}) = \left\{\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T}) :\| T_k(x) - T(x) \| \to 0, \text{ for all } x \in X\right\}$$

and

$$\mathcal{C}_0(\mathcal{T}) = \left\{\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T}) :\| T_k(x) - O(x) \| \to 0 \text{ for all } x \in X\right\} \text{ (c.f.[67])}$$

Then, $\mathcal{C}(\mathcal{T})$ and $\mathcal{C}_0(\mathcal{T})$ are normed spaces with norm defined as above in (1.3).

Remark 4.1.4. $\mathcal{C}_0(\mathcal{T}) \subset \mathcal{C}(\mathcal{T}) \subset \mathcal{B}_\infty(\mathcal{T})$.

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast [16] and also independently by Buck [8] and Schoenberg [98] for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [18], Šalát [94], Tripathy [103] and many others.

Henceforth, in this paper, without loss of generality, we considered the operators T_k for each $k \in \mathbb{N}$ from the normed space $X = \mathbb{R}$ into the normed space $Y = \mathbb{R}$ over the field $\mathbb{R}$ and x is an element of the real space $\mathbb{R}$.

Definition 4.1.5. A sequence $\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T}) \subset \mathcal{L}(\mathcal{T})$ is said to be statistically convergent to an operator T, if for every $\epsilon > 0$, we have

$$\lim_k \frac{1}{k}\left|\left\{n \in \mathbb{N} :\| T_n(x) - T(x) \| \geq \epsilon,\ n \leq k\right\}\right| = 0, \tag{4.1.5}$$

where vertical lines denote the cardinality of the enclosed set.
That is, if $\delta(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{k \in \mathbb{N} :\| T_k(x) - T(x) \| \geq \epsilon\right\}.$$

The notion of ideal convergence ($\mathcal{I}$-convergence) was introduced and studied by Kostyrko, Mačaj, Salát and Wilczyński [63],[65]. Later on, it was studied by

Šalát, Tripathy and Ziman [95],[96], Tripathy and Hazarika [101],[102], Khan *et al* [45],[43],[51] and many others.

Here we give some preliminaries about the notion of $\mathcal{I}$-convergence.

Definition 4.1.6. Let $\mathbb{N}$ be a non empty set. Then a family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal, if
1) $\mathcal{I}$ is additive. That is, $\forall A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.
2) $\mathcal{I}$ is hereditary, That is, $\forall A \in \mathcal{I}$and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 4.1.7. A non-empty family of sets $\mathcal{L}(\mathcal{I}) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$, if
1) $\phi \notin \mathcal{L}(\mathcal{I})$,
2) $\forall\ A,\ B \in \mathcal{L}(\mathcal{I}) \Rightarrow A \cap B \in \mathcal{L}(\mathcal{I})$,
3) $\forall\ A \in \mathcal{L}(\mathcal{I})$ and $A \subseteq B \Rightarrow B \in \mathcal{L}(\mathcal{I})$.

Definition 4.1.8. An Ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial, if $\mathcal{I} \neq 2^{\mathbb{N}}$.

Definition 4.1.9. A non-trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called admissible, if

$$\{\{x\} : x \in \mathbb{N}\} \subseteq \mathcal{I}.$$

Definition 4.1.10. A non-trivial ideal $\mathcal{I}$ is maximal, if there cannot exist any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

Remark 4.1.11. For each ideal $\mathcal{I}$, there is a filter $\mathcal{L}(\mathcal{I})$ corresponding to $\mathcal{I}$.
That is, $\mathcal{L}(\mathcal{I}) = \{K \subseteq \mathbb{N} : K^c \in \mathcal{I}\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 4.1.12. A sequence $\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T}) \subset \mathcal{L}(\mathcal{T})$ is said to be $\mathcal{I}$-convergent to an operator T, if for every $\epsilon > 0$, $\{k \in \mathbb{N} : \| T_k(x) - T(x) \| \geq \epsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim T_k = T$.

Definition 4.1.13. A sequence $\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T})$ is said to be I-null, if $T = O$. In this case, we write $\mathcal{I} - \lim T_k = O$.

Definition 4.1.14. A sequence $\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T})$ is said to be $\mathcal{I}$-Cauchy, if for every $\epsilon > 0$, there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : \| T_k(x) - T_m(x) \| \geq \epsilon\} \in \mathcal{I}$.

Definition 4.1.15. A sequence $\mathcal{T} = (T_k) \in \mathcal{B}_\infty(\mathcal{T})$ is said to be $\mathcal{I}$-bounded, if there exists some $M > 0$ such that $\{k \in \mathbb{N} : \| T_k(x) \| \geq M\} \in \mathcal{I}$.

Definition 4.1.16. A sequence space $E^{\mathcal{T}}$(space of operators) is said to be solid(normal), if $(\alpha_k T_k) \in E^{\mathcal{T}}$, whenever $(T_k) \in E^{\mathcal{T}}$ and for any sequence (α_k) of scalars with $| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$.

Definition 4.1.17. A sequence space $E^{\mathcal{T}}$ is said to be symmetric, if $(T_{\pi(k)}) \in E^{\mathcal{T}}$, whenever $T_k \in E^{\mathcal{T}}$, where π is a permutation on $\mathbb{N}$

Definition 4.1.18. A sequence space $E^{\mathcal{T}}$ is said to be sequence algebra, if $(T_k) * (S_k) = (T_k.S_k) \in E^{\mathcal{T}}$ whenever (T_k), $(S_k) \in E^{\mathcal{T}}$.

Definition 4.1.19. A sequence space $E^{\mathcal{T}}$ is said to be convergence free, if $(S_k) \in E^{\mathcal{T}}$, whenever $(T_k) \in E^{\mathcal{T}}$ and $T_k = O$ implies $S_k = O$, for all k.

Definition 4.1.20. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and $E^{\mathcal{T}}$ be a sequence space. A K-step space of $E^{\mathcal{T}}$ is a sequence space $\lambda_K^{E^{\mathcal{T}}} = \{(T_{k_n}) \in \mathcal{L}(\mathcal{T}) : (T_k) \in E^{\mathcal{T}}\}$.

Definition 4.1.21. A canonical pre-image of a sequence $(T_{k_n}) \in \lambda_K^{E^{\mathcal{T}}}$ is a sequence $(S_k) \in \mathcal{L}(\mathcal{T})$ defined by

$$S_k = \begin{cases} T_k, & \text{if } k \in K, \\ O, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda_K^{E^{\mathcal{T}}}$ is a set of preimages all elements in $\lambda_K^{E^{\mathcal{T}}}$. That is, $\mathcal{S}$ is in the canonical preimage of $\lambda_K^{E^{\mathcal{T}}}$ iff $\mathcal{S}$ is the canonical preimage of some $\mathcal{T} \in \lambda_K^{E^{\mathcal{T}}}$.

Definition 4.1.22. A sequence space E^T is said to be monotone, if it contains the canonical preimages of its step space.

Definition 4.1.23. A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t,\ u \geq 0$,
(3) f is increasing, and
(4) f is continuous from the right at zero.

A modulus function f is said to satisfy Δ_2 – Condition for all values of u if there exists a constant $K > 0$ such that $f(Lu) \leq \text{KL}f(u)$ for all values of $L > 1$.

The idea of modulus function was introduced by Nakano in 1953(See[79], $Nakano$, 1953).

Ruckle used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK-space and Ruckle [90], [92] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus, Ruckle [90],[92] proved that, for any modulus f,

$$X(f) \subset \ell_1 \text{ and } X(f)^{\alpha} = \ell_{\infty},$$

where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch [22]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces which are locally convex have been frequently studied by D.J.H Garling [21], G.Köthe [66] and W.H.Ruckle [90],[91],[92].

For any modulus function f, we have the inequalities

$$| f(x) - f(y) | \leq f(| x - y |)$$

and

$$f(nx) \leq nf(x), \quad \text{for all } x,\ y \in [0, \infty].$$

Definition 4.1.24. Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x,\ y \in X$,

(P_1) $g(x) = 0 \; if \; x = \theta$,

(P_2) $g(-x) = g(x)$,

(P_3) $g(x + y) \leq g(x) + g(y)$,

(P_4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

We need the following popular inequalities (see [16]) throughout the paper.

Let $p = (p_k)$ be the bounded sequence of positive reals numbers. For any complex λ, whenever $H = \sup_k(p_k) < \infty$, we have

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$$

Also, whenever $H = \sup_k(p_k)$, we have

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}),$$

where $C = \max(1, 2^{H-1})$.

We used the following lemmas for establishing some results of this article.

Lemma(I). Every solid space is monotone.

Lemma(II). Let $K \in \mathcal{L}(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$, then $M \cap K \notin \mathcal{I}$.

Lemma(III). If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$, then $M \cap \mathbb{N} \notin \mathcal{I}$.

Throughout the article $\mathcal{C}^{\mathcal{I}}(\mathcal{T})$, $\mathcal{C}_0^{\mathcal{I}}(\mathcal{T})$, $\mathcal{B}_{\infty}^{\mathcal{I}}(\mathcal{T})$, $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T})$ and $\mathcal{M}_{\mathcal{C}_0}^{\mathcal{I}}(\mathcal{T})$ are considered as the classes of all $\mathcal{I}$-convergent, $\mathcal{I}$-null, $\mathcal{I}$-bounded, bounded $\mathcal{I}$-convergent and bounded $\mathcal{I}$-null sequences of bounded linear operators, respectively.

4.2. MAIN RESULTS

In this article, we introduce the following classes of sequences.

$$\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)=\Big\{\mathcal{T}=(T_k)\in\mathcal{B}_{\infty}(\mathcal{T}):\{k\in\mathbb{N}:f(\parallel T_k(x)-L\parallel)^{p_k}\geq\epsilon\}\in\mathcal{I},\text{for some }L\Big\}; \tag{4.2.1}$$

$$\mathcal{C}_0^{\mathcal{I}}(\mathcal{T},f,p)=\Big\{\mathcal{T}=(T_k)\in\mathcal{B}_{\infty}(\mathcal{T}):\{k\in\mathbb{N}:f(\parallel T(x_k)\parallel)^{p_k}\geq\epsilon\}\in\mathcal{I}\Big\}; \tag{4.2.2}$$

$$\mathcal{B}_{\infty}^{\mathcal{I}}(\mathcal{T},f,p)=\Big\{\mathcal{T}=(T_k)\in\mathcal{B}_{\infty}(\mathcal{T}):\{k\in\mathbb{N}:\exists K>0,f(\parallel T(x_k)\parallel)^{p_k}\geq K\}\in\mathcal{I}\Big\}; \tag{4.2.3}$$

$$\mathcal{B}_{\infty}(\mathcal{T},f,p)=\Big\{x=(x_k)\in\ell_{\infty}:\sup_k f(\parallel T_k(x)\parallel)^{p_k}<\infty,\Big\}. \tag{4.2.4}$$

We also denote

$\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T},f,p)=\mathcal{B}_{\infty}(\mathcal{T},f,p)\cap\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)$ and $\mathcal{M}_{\mathcal{C}_0}^{\mathcal{I}}(\mathcal{T},f,p)=\mathcal{B}_{\infty}(\mathcal{T},f,p)\cap\mathcal{C}_0^{\mathcal{I}}(\mathcal{T},f,p)$, where $p=(p_k)$ is a bounded sequence of positive real numbers and f is a modulus function.

Theorem 4.2.1. The classes of sequences $\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)$, $\mathcal{C}_0^{\mathcal{I}}(\mathcal{T},f,p)$, $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T},f,p)$ and $\mathcal{M}_{\mathcal{C}_0}^{\mathcal{I}}(\mathcal{T},f,p)$ are linear spaces.

Proof. We shall prove the result for $\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)$. The proof for the other spaces will follow similarly.
For, let $\mathcal{T}=(T_k)$, $\mathcal{S}=(S_k)\in\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)$ and α,β be scalars.
Now, since (T_k), $(S_k)\in\mathcal{C}^{\mathcal{I}}(\mathcal{T},f,p)$. Then, for any given $\epsilon>0$, there exists some L_1, L_2 such that

$$\Big\{k\in\mathbb{N}:f(\parallel T_k(x)-L_1\parallel)^{p_k}\geq\frac{\epsilon}{2M_1}\Big\}\in\mathcal{I} \tag{4.2.5}$$

and

$$\left\{k \in \mathbb{N} : f\big(\parallel S_k(x) - L_2 \parallel \big)^{p_k} \geq \frac{\epsilon}{2M_2}, \right\} \in \mathcal{I} \tag{4.2.6}$$

where

$$M_1 = D.\max\left\{1, \sup_k |\alpha|^{p_k}\right\}$$

$$M_2 = D.\max\left\{1, \sup_k |\beta|^{p_k}\right\}$$

and

$$D = \max\{1, 2^{H-1}\} \text{ where } H = \sup_k p_k \geq 0.$$

Let

$$A_1 = \left\{k \in \mathbb{N} : f\big(\parallel T_k(x) - L_1 \parallel \big)^{p_k} < \frac{\epsilon}{2M_1}\right\} \in \mathcal{L}(\mathcal{I}) \tag{4.2.7}$$

and

$$A_2 = \left\{k \in \mathbb{N} : f\big(\parallel S_k(x) - L_2 \parallel \big)^{p_k} < \frac{\epsilon}{2M_2}\right\} \in \mathcal{L}(\mathcal{I}) \tag{4.2.8}$$

be such that $A_1^c, A_2^c \in \mathcal{I}$.
Then,

$$\begin{aligned} A_3 = &\left\{k \in \mathbb{N} : f\big(\parallel (\alpha T_k(x) + \beta S_k(x)) - (\alpha L_1 + \beta L_2) \parallel \big)^{p_k} < \epsilon\right\} \\ \supseteq &\left[\left\{k \in \mathbb{N} : |\alpha|^{p_k} f\big(\parallel T_k(x) - L_1 \parallel \big)^{p_k} < \frac{\epsilon}{2M_1} |\alpha|^{p_k}.D\right\}\right. \\ &\left.\cap \left\{k \in \mathbb{N} : |\beta|^{p_k} f\big(\parallel S_k(x) - L_2 \parallel \big)^{p_k} < \frac{\epsilon}{2M_2} |\beta|^{p_k}.D\right\}\right] \end{aligned} \tag{4.2.9}$$

implies that $A_3 \in \mathcal{L}(\mathcal{I})$.
Thus, $A_3^c = A_1^c \cup A_2^c \in \mathcal{I}$. Therefore, $\alpha(T_k) + \beta(S_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$, for all scalars α, β, and (T_k), $(S_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$.
Hence $\mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$ is a linear space.

Theorem 4.2.2. The classes of sequences $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T}, f, p)$ and $\mathcal{M}_{\mathcal{C}_0}^{\mathcal{I}}(\mathcal{T}, f, p)$ are paranormed spaces, paranormed by

$$g(\mathcal{T}) = g(T_k) = \sup_k f\big(\parallel T_k(x) \parallel \big)^{\frac{p_k}{M}}, \text{ where } M = \max\{1, \sup_k p_k\}.$$

Proof. Let $\mathcal{T} = (T_k)$, $\mathcal{S} = (S_k) \in \mathcal{M}_{\mathcal{S}}^{\mathcal{I}}(\mathcal{T}, f, p)$.
(P_1) It is Clear that $g(\mathcal{T}) = 0$ if and only if $\mathcal{T} = O$.

(P_2) $g(\mathcal{T}) = g(-\mathcal{T})$ is obvious.

(P_3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$g(\mathcal{T} + \mathcal{S}) = g(T_k + S_k) = \sup_k f\big(\| T_k + S_k)(x) \| \big)^{\frac{p_k}{M}}$$

$$= \sup_k f\big(\| T_k(x) + S_k(x) \| \big)^{\frac{p_k}{M}} \leq \sup_k f\big(\| T_k(x) \| \big)^{\frac{p_k}{M}}$$

$$+ \sup_k f\big(\| S_k(x)) \|^{\frac{p_k}{M}} = g(T_k) + g(S_k) = g(\mathcal{T}) + g(\mathcal{S}).$$

Therefore, $g(\mathcal{T} + \mathcal{S}) \leq g(\mathcal{T}) + g(\mathcal{S})$.

(P_4) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda$ $(k \to \infty)$ and (T_k), $T \in \mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T}, f, p)$ such that

$$T_k \to T \ (k \to \infty)$$

in the sense that

$$g(T_k - T) \to 0 \ (k \to \infty).$$

Then, since the inequality

$$g(T_k) \leq g(T_k - T) + g(T)$$

holds by subadditivity of g, the sequence $\{g(T_k)\}$ is bounded.

Therefore,

$$\begin{aligned} g\big[(\lambda_k T_k - \lambda T)\big] &= g\big[(\lambda_k T_k - \lambda T_k + \lambda T_k - \lambda T)\big] \\ &= g\big[(\lambda_k - \lambda)T_k + \lambda(T_k - T)\big] \\ &\leq g\big[(\lambda_k - \lambda)T_k\big] + g\big[\lambda(T_k - T)\big] \\ &\leq | (\lambda_k - \lambda) |^{\frac{p_k}{M}} g(T_k) + | \lambda |^{\frac{p_k}{M}} g(T_k - T) \to 0 \end{aligned}$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous. Hence $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T}, f, p)$ is a paranormed space.

For $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}_0}(\mathcal{T}, f, p)$, the result is similar.

Theorem 4.2.3. The space $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p)$ is closed subspace of $\mathcal{B}_{\infty}(\mathcal{T}, f, p)$.

Proof. Let $(T_k^{(n)})$ be a Cauchy sequence in $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p)$ such that $T_k^{(n)} \to T$.
We show that $T \in \mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p)$
Since $(T_k^{(n)}) \in \mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p)$. Then, there exists A_n such that

$$\{k \in \mathbb{N} : f(\parallel T_k^{(n)}(x) - A_n \parallel)^{p_k} \geq \epsilon\} \in \mathcal{I}.$$

We need to show that
(1) (A_n) converges to A.
(2) If $U = \{k \in \mathbb{N} : f(\parallel T_k(x) - A \parallel)^{p_k} < \epsilon\}$, then $U^c \in \mathcal{I}$.
(1) Since $(T_k^{(n)})$ is Cauchy sequence in $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p) \Rightarrow$ for any given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_k f(\parallel T_k^{(n)}(x) - T_k^{(q)}(x) \parallel)^{\frac{p_k}{M}} < \frac{\epsilon}{3}$, for all $n,\ q \geq k_0$.
For $\epsilon > 0$, we have

$$B_{nq} = \left\{k \in \mathbb{N} : f(\parallel T_k^{(n)}(x) - T_k^{(q)}(x) \parallel)^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\},$$

$$B_q = \left\{k \in \mathbb{N} : f(\parallel T_k^{(q)}(x) - A_q \parallel)^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\},$$

$$B_n = \left\{k \in \mathbb{N} : f(\parallel T_k^{(n)}(x) - A_n \parallel)^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}.$$

Then, $B^c_{nq}, B^c_q, B^c_n \in \mathcal{I}$
Let $B^c = B^c_{nq} \cup B^c_q \cup B^c_n$, where $B = \{k \in \mathbb{N} : f(\parallel A_q - A_n \parallel)^{p_k} < \epsilon\}$. Then $B^c \in \mathcal{I}$.
We choose $k_0 \in B^c$. Then for each $n,\ q \geq k_0$, we have

$$\left\{k \in \mathbb{N} : f(\parallel A_q - A_n \parallel)^{p_k} < \epsilon\right\} \supseteq \left[\left\{k \in \mathbb{N} : f(\parallel A_q - T_k^{(q)}(x) \parallel)^{p_k} < \left(\tfrac{\epsilon}{3}\right)^M\right\}\right.$$

$$\cap \left\{k \in \mathbb{N} : f(\parallel T_k^{(q)}(x) - T_k^{(n)}(x) \parallel)^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}$$

$$\cap \left\{k \in \mathbb{N} : f(\parallel T_k^{(n)}(x) - A_n \parallel)^{p_k} < \left(\frac{\epsilon}{3}\right)^M \right\}\Big].$$

Then, (A_n) is a Cauchy sequence of vectors in Y and Y is complete, so there exists some A in Y such that $A_n \to A$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then, we show that, if

$$U = \{k \in \mathbb{N} : f(\parallel T_k(x) - A \parallel)^{p_k} < \delta\}.$$

Then, $U^c \in \mathcal{I}$.

Since $(T_k^{(n)}) \to T$ then, there exists $q_0 \in \mathbb{N}$ such that

$$P = \left\{k \in \mathbb{N} : f(\parallel T_k^{(q_0)}(x) - T_k(x) \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\} \tag{4.2.10}$$

implies $P^c \in \mathcal{I}$.

The number q_0 can be chosen that together with (4.2.10), we have

$$Q = \left\{k \in \mathbb{N} : f(\parallel A_{q_0} - A \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\}$$

such that $Q^c \in \mathcal{I}$.

Since $\{k \in \mathbb{N} : f(\parallel T_k^{(q_0)}(x) - A_{q_0} \parallel)^{p_k} \geq \delta\} \in \mathcal{I}$. Then, we have a subset S of $\mathbb{N}$ such that $S^c \in \mathcal{I}$, where $S = \left\{k \in \mathbb{N} : f(\parallel T_k^{(q_0)}(x) - A_{q_0} \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\}$.

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{k \in \mathbb{N} : f(\parallel T_k(x) - A \parallel)^{p_k} < \delta\}$.

Therefore, for each $k \in U^c$, we have

$$\{k \in \mathbb{N} : f(\parallel T_k(x) - A \parallel)^{p_k} < \delta\}$$

$$\supseteq \Big[\left\{k \in \mathbb{N} : f(\parallel T_k(x) - T_k^{(q_0)}(x) \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\}$$

$$\cap \left\{k \in \mathbb{N} : f(\parallel T_k^{(q_0)}(x) - A_{q_0} \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\}$$

$$\cap \left\{k \in \mathbb{N} : f(\parallel A_{q_0} - A \parallel)^{p_k} < \left(\frac{\delta}{3D}\right)^M \right\}\Big]. \tag{4.2.11}$$

Then, the result follows from (4.2.11).

Since, the inclusions $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}(\mathcal{T}, f, p) \subset \mathcal{B}_\infty(\mathcal{T}, f, p)$ and $\mathcal{M}_{\mathcal{C}_0}^{\mathcal{I}}(\mathcal{T}, f, p) \subset \mathcal{B}_\infty(\mathcal{T}, f, p)$ are strict so in view of Theorem (4.2.3) we have the following result.

Theorem 4.2.4. The spaces $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}}(\mathcal{T}, f, p)$ and $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}_0}(\mathcal{T}, f, p)$ are nowhere dense subsets of $\mathcal{B}_\infty(\mathcal{T}, f, p)$.

Theorem 4.2.5. The spaces $\mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$ and $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}_0}(\mathcal{T}, f, p)$ are both solid and monotone.

Proof. We shall prove the result for $\mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$. For $\mathcal{M}^{\mathcal{I}}_{\mathcal{C}_0}(\mathcal{T}, f, p)$ the result follows similarly.
For, let $\mathcal{T} = (T_k) \in \mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$ and (α_k) be a sequence of scalars with $| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$.
Since $| \alpha_k |^{p_k} \leq \max\{1, | \alpha_k |^G\} \leq 1$, for all $k \in \mathbb{N}$,
we have

$$f(\| \alpha_k T_k(x) \|)^{p_k} \leq f(\| T_k(x) \|)^{p_k}, \text{ for all k} \in \mathbb{N}.$$

which further implies that

$$\{k \in \mathbb{N} : f(\| T_k(x) \|)^{p_k} \geq \epsilon\} \supseteq \{k \in \mathbb{N} : f(\| \alpha_k T_k(x) \|)^{p_k} \geq \epsilon\}.$$

Thus, $\alpha_k(T_k) \in \mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$.
Therefore, the space $\mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$ is solid and hence by lemma(I), it is monotone.

Theorem 4.2.6. Let $G = \sup_k p_k < \infty$ and $\mathcal{I}$ be an admissible ideal. Then, the following are equivalent;
(a) $(T_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$;
(b) there exists $(S_k) \in \mathcal{C}(\mathcal{T}, f, p)$ such that $T_k = S_k$, for a.a.k.r. $\mathcal{I}$;
(c) there exists $(S_k) \in \mathcal{C}(\mathcal{T}, f, p)$ and $U_k \in \mathcal{C}^{\mathcal{I}}_0(\mathcal{T}, f, p)$ such that $T_k = S_k + U_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f(\| S_k(x) - L \|)^{p_k} \geq \epsilon\} \in \mathcal{I}$;
(d) there exists a subset $K = \{k_1 < k_2 < k_3 < k_4 ...\}$ of $\mathbb{N}$ such that $K \in \mathcal{L}(\mathcal{I})$ and $\lim_{n\to\infty} f(\| T_{k_n}(x) - L \|)^{p_{k_n}} = 0$.

Proof. (a) implies (b).
Let $\mathcal{T} = (T_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$. Then, there exists some L such that

$$\{k \in \mathbb{N} : f(\| T_k(x) - L \|)^{p_k} \geq \epsilon\} \in \mathcal{I}.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : f(\parallel T_k(x) - L \parallel)^{p_k} \geq t^{-1}\} \in \mathcal{I}.$$

Define a sequence (S_k) as

$$S_k = T_k, \quad \text{for all k} \leq m_1.$$

For $m_t < k \leq m_{t+1}, \ t \in \mathbb{N}$

$$S_k = \begin{cases} T_k, \text{ if } f(\parallel T_k(x) - L \parallel)^{p_k} < t^{-1}, \\ \qquad L, \text{otherwise.} \end{cases}$$

Then, $(S_k) \in \mathcal{C}(\mathcal{T}, f, p)$ and from the inclusion

$$\{k \leq m_t : T_k \neq S_k\} \subseteq \{k \leq m_t : f(\parallel T_k(x) - L \parallel)^{p_k} \geq \epsilon\} \in \mathcal{I}.$$

we get $T_k = S_k$ for a.a.k.r. $\mathcal{I}$.
(b) implies (c). For $\mathcal{T} = (T_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$, then, there exists $(S_k) \in \mathcal{C}(\mathcal{T}, f, p)$ such that $T_k = S_k$, for a.a.k.r. $\mathcal{I}$. Let $K = \{k \in \mathbb{N} : T_k \neq S_k\}$, then, $K \in \mathcal{I}$.
Define U_k as follows.

$$U_k = \begin{cases} T_k - S_k, \text{if } k \in K, \\ \quad O, \ \text{ if } k \notin K. \end{cases}$$

Then, $U_k \in \mathcal{C}_0^{\mathcal{I}}(\mathcal{T}, f, p)$ and $S_k \in \mathcal{C}(\mathcal{T}, f, p)$.
(c) implies (d). Suppose (c) holds. Let $\epsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : f(\parallel U_k(x) \parallel)^{p_k} \geq \epsilon\} \in \mathcal{I}$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < k_4 ...\} \in \mathcal{L}(\mathcal{I}).$$

Then, we have

$$\lim_{k \to \infty} f(\parallel T_{k_n}(x) - L \parallel)^{p_{k_n}} = 0.$$

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < k_4 ...\} \in \mathcal{L}(\mathcal{I})$ and

$$\lim_{k \to \infty} f(\parallel T_{k_n}(x) - L \parallel)^{p_{k_n}} = 0.$$

Then, for any $\epsilon > 0$, and Lemma (II), we have

$$\{k \in \mathbb{N} : f(\parallel T_k(x) - L \parallel)^{p_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : f(\parallel T_k(x) - L \parallel)^{p_k} \geq \epsilon\}.$$

Thus, $(T_k) \in \mathcal{C}^{\mathcal{I}}(\mathcal{T}, f, p)$.

Theorem 4.2.7. Let f_1 and f_2 be two modulus functions satisfying Δ_2-Condition and $p = (p_k) \in \ell_\infty$ be a sequence of positive real numbers. Then,
(a) $\mathcal{X}(\mathcal{T}, f_2, p) \subseteq \mathcal{X}(\mathcal{T}, f_1 f_2, p)$,
(b) $\mathcal{X}(\mathcal{T}, f_1, p) \cap (\mathcal{T}, f_2, p) \subseteq \mathcal{X}(\mathcal{T}, f_1 + f_2, p)$,
for $\mathcal{X} = \mathcal{C}^{\mathcal{I}}$, $\mathcal{C}_\circ^{\mathcal{I}}$, $\mathcal{M}_{\mathcal{C}}^{\mathcal{I}}$ and $\mathcal{M}_{\mathcal{C}_\circ}^{\mathcal{I}}$.

Proof.(a) Let $\mathcal{T} = (T_k) \in \mathcal{C}_\circ^{\mathcal{I}}(\mathcal{T}, f_2, p)$ be any arbitrary element. Then,

$$\left\{k \in \mathbb{N} : f_2\Big(\parallel T_k(x) \parallel\Big)^{p_k} \geq \epsilon\right\} \in \mathcal{I}. \tag{4.2.12}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_1(t) < \epsilon,\ 0 \leq t \leq \delta$.
Let us denote

$$S_k = f_2\Big(\parallel T_k(x) \parallel\Big) \tag{4.2.13}$$

and consider

$$\lim_k f_1\big(S_k\big)^{p_k} = \lim_{S_k \leq \delta, k \in \mathbb{N}} f_1\big(S_k\big)^{p_k} + \lim_{S_k > \delta, k \in \mathbb{N}} f_1\big(S_k\big)^{p_k}.$$

Now, since f_1 is an modulus function , we have

$$\lim_{S_k \leq \delta, k \in \mathbb{N}} f_1\big(S_k\big)^{p_k} \leq f_1\big(2\big)^{p_k} \lim_{S_k \leq \delta, k \in \mathbb{N}} (S_k)^{p_k}. \tag{4.2.14}$$

For $S_k > \delta$, we have

$$S_k < \frac{S_k}{\delta} < 1 + \frac{S_k}{\delta}.$$

Now, since f_1 is non-decreasing and modulus, it follows that

$$f_1(S_k) < f_1(1 + \frac{S_k}{\delta}) < \frac{1}{2} f_1(2) + \frac{1}{2} f_1(\frac{2S_k}{\delta}).$$

Again, since f_1 satisfies $\Delta_2 -$ Condition, we have

$$f_1\big(S_k\big) < \frac{1}{2} K \frac{(S_k)}{\delta} f_1(2) + \frac{1}{2} K \frac{(S_k)}{\delta} f_1(2).$$

Thus, $f_1(S_k) < K\frac{(S_k)}{\delta} f_1(2)$.
Hence,

$$\lim_{S_k>\delta, k\in\mathbb{N}} f_1(S_k)^{p_k} \leq \max\{1, (K\delta^{-1}f_1(2))^H\} \lim_{S_k>\delta, k\in\mathbb{N}} (S_k)^{p_k}, \ H = \max\{1, \sup_k p_k\}. \tag{4.2.15}$$

Therefore, from (4.2.13), (4.2.14) and (4.2.15), we have
$(T_k) \in \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1 f_2, p)$
Thus, $\mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_2, p) \subseteq \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1 f_2, p)$. Hence, $\mathcal{X}(\mathcal{T}, f_2, p) \subseteq \mathcal{X}(\mathcal{T}, f_1 f_2, p)$ for $\mathcal{X} = \mathcal{C}_\circ^\mathcal{I}$.
For $\mathcal{X} = \mathcal{C}^\mathcal{I}$, $\mathcal{M}_\mathcal{C}^\mathcal{I}$ and $\mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}$ the inclusions can be established similarly.
(b). Let $\mathcal{T} = (T_k) \in \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1, p) \cap \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_2, p)$. Let $\epsilon > 0$ be given. Then,

$$\left\{k \in \mathbb{N} : f_1\Big(\| T_k(x) \| \Big)^{p_k} \geq \epsilon\right\} \in \mathcal{I} \tag{4.2.16}$$

and

$$\left\{k \in \mathbb{N} : f_2\Big(\| T_k(x) \| \Big)^{p_k} \geq \epsilon\right\} \in \mathcal{I} \tag{4.2.17}$$

Therefore, from (4.2.16) and (4.2.17)

$$\left\{k \in \mathbb{N} : (f_1 + f_2)\Big(\| T_k(x) \| \Big)^{p_k} \geq \epsilon\right\} \in \mathcal{I}$$

Thus, $\mathcal{T} = (T_k) \in \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1 + f_2, p)$
Hence, $\mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1, p) \cap \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_2, p) \subseteq \mathcal{C}_\circ^\mathcal{I}(\mathcal{T}, f_1 + f_2, p)$
For $\mathcal{X} = \mathcal{C}^\mathcal{I}$, $\mathcal{M}_\mathcal{C}^\mathcal{I}$ and $\mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}$, the inclusions are similar.

For $f_2(x) = x$ and $f_1(x) = f(x)$, $\forall\, x \in [0, \infty)$, we have the following corollary.

Corollary 4.2.8. $\mathcal{X}(\mathcal{T}, p) \subseteq \mathcal{X}(\mathcal{T}, f, p)$ for $\mathcal{X} = \mathcal{C}^\mathcal{I}$, $\mathcal{C}_\circ^\mathcal{I}$, $\mathcal{M}_\mathcal{C}^\mathcal{I}$ and $\mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}$

Theorem 4.2.9. Let (p_k) and (q_k) be two sequences of positive real numbers. Then, $\mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}(\mathcal{T}, f, p) \supseteq \mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}(\mathcal{T}, f, q)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \mathcal{L}(\mathcal{I})$.

Proof. Let $\liminf_{k\in K} \frac{p_k}{q_k} > 0$ and $(T_k) \in \mathcal{M}_{\mathcal{C}_\circ}^\mathcal{I}(\mathcal{T}, f, q)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$ for sufficiently large $k \in K$.

Since, $(T_k) \in \mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, q)$. For a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : f\big(\parallel T_k(x) \parallel \big)^{q_k} \geq \epsilon\} \in \mathcal{I}.$$

Let $G_0 = K^c \cup B_0$. Then $G_0 \in \mathcal{I}$. Then, for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : f\big(\parallel T_k(x) \parallel \big)^{p_k} \geq \epsilon\} \subseteq \{k \in \mathbb{N} : f\big(\parallel T_k(x) \parallel \big)^{\beta q_k} \geq \epsilon\} \in \mathcal{I}.$$

Therefore, $(T_k) \in \mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, p)$.
The converse part of the result follows obviously.

Theorem 4.2.10. Let (p_k) and (q_k) be two sequences of positive real numbers. Then, $\mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, q) \supseteq \mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, p)$ if and only if $\liminf\limits_{k \in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \mathcal{L}(\mathcal{I})$.

Proof. The proof is similar to the proof Theorem (4.2.9).

Theorem 4.2.11. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, q) = \mathcal{M}^{\mathcal{I}}_{C_o}(\mathcal{T}, f, p)$ if and only if $\liminf\limits_{k \in K} \frac{p_k}{q_k} > 0$ and $\liminf\limits_{k \in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \mathcal{L}(\mathcal{I})$.

Proof. On combining Theorems (4.2.9) and (4.2.10), we get the desired result.

CHAPTER 5

ON SOME I-CONVERGENT SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND AN ORLICZ FUNCTION

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CHAPTER-5

ON SOME I-CONVERGENT SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND AN ORLICZ FUNCTION

5.1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Definition 5.1.1. Let K be a non-trivial valued field and X be a vector space over K. Then a real valued mapping $\| \cdot \|$ on X is said to be a norm on or over X if it satisfies the following properties.

1) $\| x \| \geq 0$, $\| x \| = 0 \Leftrightarrow x = 0$

2) $\| \alpha x \| = | \alpha | \| x \|$

3) $\| x + y \| \leq \| x \| + \| y \|$, for all $\alpha \in K$, $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a normed linear space over K.

Definition 5.1.2. A linear operator T is an operator such that

1) the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field,

2) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all, $x, y \in \mathcal{D}(T)$.

Definition 5.1.3. Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator T is said to be bounded if there exists a real k such that

$$\| Tx \| \leq k \| x \|, \text{ for all, } x \in \mathcal{D}(T).$$

''Mathematics knows no races or geographic boundaries, for mathematics, the cultural world is one country."-*David Hilbert*

The set of all bounded linear operators $\mathcal{B}(X,Y)$ is a normed linear space normed by

$$\| T \| = \sup_{x \in X, \|x\|=1} \| Tx \| \qquad (see[67])$$

and $\mathcal{B}(X,Y)$ is a Banach space if Y is Banach space.

Definition 5.1.4. Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if

1). T is linear,

2). T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X,Y)$ is closed subspace of $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$ is a Banach space if Y is Banach space.

Throughout the paper, we denote ℓ_∞, c and c_0 as the Banach spaces of bounded, convergent and null sequences of reals respectively with norm

$$\|x\| = \sup_k | x_k | .$$

Following Basar and Altay[5] and Sengönül[99], we introduce the sequence spaces $\mathcal{S}$ and $\mathcal{S}_0$ with the help of compact operator T on $\mathbb{R}$ as follows.

$$\mathcal{S} = \{x = (x_k) \in \ell_\infty : T(x) \in c\}$$

and

$$\mathcal{S}_0 = \{x = (x_k) \in \ell_\infty : T(x) \in c_0\}.$$

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast[16] and also independently by Buck[8] and Schoenberg[98] for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy[18], Šalát[94], Tripathy[103] and the references therein.

Definition 5.1.5. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have

$$\lim_n \frac{1}{n}|\{k \in \mathbb{N} : |x_k - L| \geq \epsilon, k \leq n\}| = 0.$$

where vertical lines denote the cardinality of the enclosed set.
That is, if $\delta(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{k \in \mathbb{N} :| x_k - L |\geq \epsilon\right\}$$

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko *et al* [64],[65]. Later on, it was studied by Šalát, Tripathy and Ziman [95],[96], Tripathy and Hazarika [101],[102], Khan and Ebadullah [43],[45] and the references therein.

Now, we recall the following definitions

Definition 5.1.6. Let N be a non empty set. Then, a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal if
1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$
2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 5.1.7. A non-empty family of sets $\mathcal{L}(I) \subseteq 2^N$ is said to be filter on N if and only if
1) $\Phi \notin \mathcal{L}$(I),
2) $\forall$ A, $B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$,
3) $\forall$ $A \in \mathcal{L}(I)$ and $A \subseteq B \Rightarrow B \in \mathcal{L}(I)$.

Definition 5.1.8. An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$.

Definition 5.1.9. A non-trivial ideal $I \subseteq 2^N$ is called admissible if

$$\{\{x\} : x \in N\} \subseteq I.$$

Definition 5.1.10. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 5.1.11. For each ideal I, there is a filter $\mathcal{L}(I)$ corresponding to I. i.e $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Definition 5.1.12. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in N : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 5.1.13. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 5.1.14. A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in N : |x_k| \geq M\} \in I$.

Definition 5.1.15. A sequence space E said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$.

Definition 5.1.16. A sequence space E said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$

Definition 5.1.17. A sequence space E said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 5.1.18. A sequence space E said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 5.1.19. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 5.1.20. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e.

y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 5.1.21. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 5.1.22. A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions
(i) M is continuous, convex and non-decreasing
(ii) $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$

Remark 5.1.23. If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark 5.1.24. If M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy $\Delta_2 -$ Condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leq \mathrm{KL}M(u)$ for all values of $L > 1$.

Lindenstrauss and Tzafriri[68] used the idea of an Orlicz function to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}.$$

The space ℓ_M becomes a Banach space with the norm

$$\| x \| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}.$$

which is called an Orlicz sequence space.The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = t^P$ for $1 < p < \infty$.

Later on, some Orlicz sequence spaces were investigated by Parashar and Choudhury[87], Tripathy and Hazarika[101], Maddox[71], Bhardwaj and Singh[6] Khan and Ebadullah[43] and the references therein.

Throughout the article, we use the similar techniques as had been used in [101], [102].

We used the following lemmas for establishing some results of this article.

Lemma (I). Every solid space is monotone

Lemma (II). If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

Lemma (III). Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Throughout the article, $\mathcal{S}_\infty$, $\mathcal{S}^I$, $\mathcal{S}_0^I$, $\mathcal{S}_\infty^I$, $\mathcal{M}_{\mathcal{S}}^I$, and $\mathcal{M}_{\mathcal{S}_0}^I$ are considered as classes of bounded, I-convergent, I-null, I-bounded, bounded I-convergent and bounded I-null sequences defined by a compact operator T on the real space $\mathbb{R}$.

5.2. MAIN RESULTS

In this article we introduce and study the following sequence spaces.

$$\mathcal{S}^I(M) = \left\{ x = (x_k) \in \ell_\infty : I\text{-}\lim_k M\left(\frac{|T(x_k) - L|}{\rho}\right) = 0, \text{for some L} \in \mathbb{C}, \rho > 0 \right\}, \tag{5.2.1}$$

$$\mathcal{S}_0^I(M) = \left\{ x = (x_k) \in \ell_\infty : I - \lim_k M\left(\frac{|T(x_k)|}{\rho}\right) = 0, \rho > 0 \right\}, \tag{5.2.2}$$

$$\mathcal{S}_\infty^I = \left\{ x = (x_k) \in \ell_\infty : \exists K > 0 \ s.t. \{k \in \mathbb{N} : M\left(\frac{|T(x_k)|}{\rho}\right) \geq K, \rho > 0\} \in I \right\} \tag{5.2.3}$$

$$\mathcal{S}_\infty(M) = \left\{ x = (x_k) \in \ell_\infty : \sup_k M\left(\frac{|T(x_k)|}{\rho}\right) < \infty, \rho > 0 \right\}. \tag{5.2.4}$$

We also denote

$\mathcal{M}_{\mathcal{S}}^I(M) = \mathcal{S}^I(M) \cap \mathcal{S}_\infty(M)$ and $\mathcal{M}_{\mathcal{S}_0}^I(M) = \mathcal{S}_0^I(M) \cap \mathcal{S}_\infty(M)$.

Theorem 5.2.1. For any Orlicz function M, the classes of sequence $\mathcal{S}_0^I(M)$, $\mathcal{S}^I(M)$, $\mathcal{M}^I_{S_0}(M)$ and $\mathcal{M}^I_S(M)$ are the linear spaces.

Proof. We shall prove the result for the space $\mathcal{S}^I(M)$. The proof of the others follow similarly.

For, let $x = (x_k),\ y = (y_k) \in \mathcal{S}^I(M)$ be any two arbitrary elements and let α, β be scalars.

Now, since $(x_k),\ (y_k) \in \mathcal{S}^I(M)\ \Rightarrow$ For any $\epsilon > 0$, there exists some +ve numbers $L_1,\ L_2 \in \mathbb{C}$ and $\rho_1,\ \rho_2 > 0$ such that the sets

$$A_1 = \left\{k \in \mathbb{N} : M\left(\frac{|T(x_k) - L_1|}{\rho_1}\right) \geq \frac{\epsilon}{2}\right\} \in I \tag{5.2.5}$$

and

$$A_2 = \left\{k \in \mathbb{N} : M\left(\frac{|T(y_k) - L_2|}{\rho_2}\right) \geq \frac{\epsilon}{2}\right\} \in I. \tag{5.2.6}$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$.

$$\tag{5.2.7}$$

Since, M is non-decreasing and convex function, we have,

$$\begin{aligned} & M\left(\frac{|(\alpha T(x_k) + \beta T(y_k)) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) \\ & \leq M\left(\frac{|\alpha||T(x_k) - L_1|}{\rho_3}\right) + M\left(\frac{|\beta||T(y_k) - L_2|}{\rho_3}\right) \\ & \leq M\left(\frac{|T(x_k) - L_1|}{\rho_1}\right) + M\left(\frac{|T(y_k) - L_2|}{\rho_2}\right). \end{aligned} \tag{5.2.8}$$

Therefore, from (5.2.5), (5.2.6) and (5.2.8), we have,

$$\left\{k \in \mathbb{N} : M\left(\frac{|(\alpha T(x_k) + \beta T(y_k)) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) \geq \epsilon\right\} \subseteq \left[A_1 \cup A_2\right] \in I$$

implies that

$$\left\{k \in \mathbb{N} : M\left(\frac{|(\alpha T(x_k) + \beta T(y_k)) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) \geq \epsilon\right\} \in I.$$

Therefore $\alpha x_k + \beta y_k \in \mathcal{S}^I(M)$

Hence $S^I(M)$ is linear

Theorem 5.2.2. For any Orlicz function M, the spaces $\mathcal{M}^I_S(M)$ and $\mathcal{M}^I_{S_\circ}(M)$ are Banach spaces normed by

$$\| x \| = \inf\left\{\rho > 0 : \sup_k M\left(\frac{| T(x_k) |}{\rho}\right) < 1, \rho > 0\right\}. \quad (see[81])$$

Proof. The proof this result can be established easily in view of existing techniques and hence omitted.

Theorem 5.2.3. Let M_1 and M_2 be two Orlicz functions and satisfying $\Delta_2 -$ Condition. Then
(a) $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1 M_2)$
(b) $\mathcal{X}(M_1) \cap (M_2) \subseteq \mathcal{X}(M_1 + M_2)$
for $\mathcal{X}= S^I$, $S^I_\circ$, $\mathcal{M}^I_S$ and $\mathcal{M}^I_{S_\circ}$

Proof.(a) Let $x = (x_k) \in S^I_\circ(M_2)$ be any arbitrary element$\Rightarrow \exists\ \rho > 0$ such that

$$I - \lim_k M_2\left(\frac{| T(x_k) |}{\rho}\right) = 0. \tag{5.2.9}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon,\ 0 \leq t \leq \delta$.
Put

$$y_k = M_2\left(\frac{| T(x_k) |}{\rho}\right)$$

and consider

$$\lim_k M_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k).$$

Now, since M_1 is an Orlicz function , we have

$$M_1(\lambda x) \leq \lambda M_1(x), \quad \text{for all } \lambda \text{ with } 0 < \lambda < 1.$$

Therefore,

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) \leq M_1(2) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k). \tag{5.2.10}$$

For $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Now, since M_1 is non-decreasing and convex, it follows that

$$M_1(y_k) < M_1(1+\frac{y_k}{\delta}) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_k}{\delta}).$$

Again, since M_1 satisfies Δ_2 -- Condition, we have

$$M_1(y_k) < \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2) + \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2).$$

Thus $M_1(y_k) < K\frac{(y_k)}{\delta}M_1(2)$
Hence

$$\lim_{y_k>\delta, k\in\mathbb{N}} M_1(y_k) \leq \max\{1, K\delta^{-1}M_1(2) \lim_{y_k>\delta, k\in\mathbb{N}}(y_k). \tag{5.2.11}$$

Therefore, from (5.2.9), (5.2.10) and (5.2.11), we have

$$I - \lim_k M_1(y_k) = 0$$

That is

$$I - \lim_k M_1 M_2\left(\frac{|T(x_k)|}{\rho}\right) = 0$$

implies that
$(x_k) \in S_\circ^I(M_1M_2)$
Thus, $S_\circ^I(M_2) \subseteq S_\circ^I(M_1M_2)$. Hence, $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1M_2)$ for $\mathcal{X}= S_\circ^I$.
For $\mathcal{X}= S^I$, $\mathcal{M}_S^I$ and $\mathcal{M}_{S_\circ}^I$ the inclusions can be established similarly.

(b). Let $x = (x_k) \in S_\circ^I(M_1) \cap S_\circ^I(M_2)$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$I - \lim_k M_1\left(\frac{|T(x_k)|}{\rho}\right) = 0$$

and

$$I - \lim_k M_2\left(\frac{|T(x_k)|}{\rho}\right) = 0$$

Therefore,

$$I - \lim_k (M_1 + M_2)\left(\frac{|T(x_k)|}{\rho}\right) = I - \lim_k M_1\left(\frac{|T(x_k)|}{\rho}\right) + I - \lim_k M_2\left(\frac{|T(x_k)|}{\rho}\right)$$

implies that

$$I - \lim_k (M_1 + M_2)\left(\frac{| T(x_k) |}{\rho}\right) = 0.$$

Thus $x = (x_k) \in \mathcal{S}^I_\circ(M_1 + M_2)$

Hence $\mathcal{S}^I_\circ(M_1) \cap \mathcal{S}^I_\circ(M_2) \subseteq \mathcal{S}^I_\circ(M_1 + M_2)$

For $\mathcal{X}= \mathcal{S}^I$, $\mathcal{M}^I_{\mathcal{S}}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$ the inclusions are similar.

For $M_2(x) = x$ and $M_1(x) = M(x)$, $\forall\ x \in [0, \infty)$, we have the following corollary.

Corollary. $\mathcal{X} \subseteq \mathcal{X}(M)$ for $\mathcal{X}= \mathcal{S}^I$, $\mathcal{S}^I_\circ$, $\mathcal{M}^I_{\mathcal{S}}$ and $\mathcal{M}^I_{\mathcal{S}_\circ}$

Theorem 5.2.4. For any orlicz function M, the spaces $\mathcal{S}^I_\circ(M)$ and $\mathcal{M}^I_{\mathcal{S}_\circ}(M)$ are solid and monotone.

Proof. We prove the result for the space $\mathcal{S}^I_\circ(M)$. For $\mathcal{M}^I_{\mathcal{S}_\circ}(M)$, the proof can be obtained similarly.

For, let $(x_k) \in \mathcal{S}^I_\circ(M)$ be any arbitrary element. $\Rightarrow\ \exists\ \rho > 0$ such that

$I - \lim_k M\left(\frac{|T(x_k)|}{\rho}\right) = 0.$

Let (α_k) be a sequence of scalars such that $| \alpha_k |\leq 1$, for all k $\in \mathbb{N}$.

Now, since M is an Orlicz function and for $\epsilon > 0$, the results follows from the following inclusion

$$\left\{k \in \mathbb{N} : M\left(\frac{| T(\alpha_k x_k) |}{\rho}\right) \geq \epsilon\right\} \subseteq \left\{k \in \mathbb{N} : M\left(\frac{| T(x_k) |}{\rho}\right) \geq \epsilon\right\}.$$

That is

$$I - \lim_k M\left(\frac{| T(\alpha_k x_k) |}{\rho}\right) = 0.$$

Thus $(\alpha_k x_k) \in \mathcal{S}^I_\circ(M)$.

Hence $\mathcal{S}^I_\circ(M)$ is solid.

Therefore, by lemma (I), $\mathcal{S}^I_\circ(M)$ is monotone. Hence the result.

Theorem 5.2.5. For any orlicz function M, the spaces $\mathcal{S}^I(M)$ and $\mathcal{M}^I_{\mathcal{S}}(M)$ are neither solid nor monotone in general.

Proof. Here we give a counter example for the establishment of this result.
For, let us consider $I = I_\delta$, $M(x) = x^2$, for all $x \in [0, \infty)$ and T an identity operator on $\mathbb{R}$.
Consider, the K-step space $B_K(M)$ of $B(M)$ as follows.
Let $(x_k) \in B(M)$ and $(y_k) \in B_K(M)$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the space (x_k) defined as $x_k = 1$, for all k $\in \mathbb{N}$. Then x_k belongs to $S^I(M)$ and $\mathcal{M}^I_S(M)$ but its K-step space pre-image does not belong to $S^I(M)$ and $\mathcal{M}^I_S(M)$. Thus $S^I(M)$ and $\mathcal{M}^I_S(M)$ are not monotone and hence by lemma (I) they are not solid.

Theorem 5.2.6. For an Orlicz function M, the spaces $S^I_\circ(M)$ and $S^I(M)$ are not convergence free

Proof Let $I = I_f$, $M(x) = x^3$ for all x $\in [0, \infty)$ and T an identity operator on $\mathbb{R}$.
Consider the sequences (x_k) and (y_k) defined as follows.

$$x_k = \frac{1}{k} \text{ and } y_k = k, \text{ for all k} \in \mathbb{N}.$$

Then, (x_k) belongs to $S^I_\circ(M)$ and $S^I(M)$ but (y_k) does not belong to $S^I_\circ(M)$ and $S^I(M)$.
Hence the spaces $S^I_\circ(M)$ and $S^I(M)$ are not convergence free.

Theorem 5.2.7. For an Orlicz function M and an identity operator T on $\mathbb{R}$, the spaces $S^I_\circ(M)$ and $S^I(M)$ are sequence algebra.

Proof. Here we consider $S^I_\circ(M)$. For the other one, result is similar.
Let $x = (x_k)$, $y = (y_k) \in S^I_\circ(M)$ be any two arbitrary elements.
$\Rightarrow \exists\ \rho_1, \rho_2 > 0$ such that

$$I - \lim_k M\left(\frac{|T(x_k)|}{\rho_1}\right) = 0$$

and

$$I - \lim_k M\left(\frac{|T(y_k)|}{\rho_2}\right) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$.
Then, it is obvious that

$$I - \lim_k M\Big(\frac{| T(x_k)T(y_k) |}{\rho}\Big) = 0$$

implies that

$$(x_k.y_k) = (x_k y_k) \in \mathcal{S}^I_\circ(M).$$

Hence, $\mathcal{S}^I_\circ(M)$ is a Sequence algebra.

Theorem 5.2.8. Let M be an Orlicz function. Then,
$\mathcal{S}^I_\circ(M) \subsetneq \mathcal{S}^I(M) \subsetneq \mathcal{S}^I_\infty(M)$.

Proof. Let M be an Orlicz function. Then, we have to show that
$\mathcal{S}^I_\circ(M) \subsetneq \mathcal{S}^I(M) \subsetneq \mathcal{S}^I_\infty(M)$
Firstly, $\mathcal{S}^I_\circ(M) \subsetneq \mathcal{S}^I(M)$ is obvious.
Now, let $x = (x_k) \in \mathcal{S}^I(M)$ be any arbitrary element
$\Rightarrow \exists\, \rho > 0$ such that $I - \lim_k M\big(\frac{|T(x_k)-L|}{\rho}\big) = 0$ for some $L \in \mathbb{C}$.
Now,
$M\big(\frac{|T(x_k)|}{2\rho}\big) \leq \frac{1}{2}M\big(\frac{|T(x_k)-L|}{\rho}\big) + \frac{1}{2}M\big(\frac{|L|}{\rho}\big)$.
Taking supremum over k to both sides, we have
$x = (x_k) \in \mathcal{S}^I_\infty(M)$.
Thus $\mathcal{S}^I(M) \subsetneq \mathcal{S}^I_\infty(M)$.
Hence $\mathcal{S}^I_\circ(M) \subsetneq \mathcal{S}^I(M) \subsetneq \mathcal{S}^I_\infty(M)$.

Theorem 5.2.9. The set $\mathcal{M}^I_S(M)$ is closed subspace of $\mathcal{S}_\infty(M)$.

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $\mathcal{M}^I_S(M)$ such that $x^{(i)} \to x$.
We show that $x \in \mathcal{M}^I_S(M)$
Since $(x_k^{(i)}) \in \mathcal{M}^I_S(M)$, then there exists a sequence a_i, and $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : M\left(\frac{| T(x_k^{(i)}) - a_i |}{\rho}\right) \geq \epsilon\right\} \in I$$

We need to show that
(1) (a_i) converges to a.

(2) If $U = \left\{k \in \mathbb{N} : M\left(\frac{|T(x_k^{(i)}) - a|}{\rho}\right) < \epsilon\right\}$, then $U^c \in I$.

(1) Since $(x_k^{(i)})$ is Cauchy sequence in $\mathcal{M}_S^I(M) \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k M\left(\frac{| T(x_k^{(i)}) - T(x_k^{(j)}) |}{\rho}\right) < \frac{\epsilon}{3}, \text{ for all i, j} \geq k_0.$$

For $\epsilon > 0$, we have

$$B_{ij} = \{k \in \mathbb{N} : M\left(\frac{| T(x_k^{(i)}) - T(x_k^{(j)}) |}{\rho}\right) < \frac{\epsilon}{3}\}$$

$$B_i = \{k \in \mathbb{N} : M\left(\frac{| T(x_k^{(i)}) - a_i |}{\rho}\right) < \frac{\epsilon}{3}\}$$

$$B_j = \{k \in \mathbb{N} : M\left(\frac{| T(x_k^{(j)}) - a_j |}{\rho}\right) < \frac{\epsilon}{3}\}$$

Then, $B_{ij}^c, B_i^c, B_j^c \in I$

Let $B^c = B_{ij}^c \cup B_i^c \cup B_j^c$, where $B = \left\{k \in \mathbb{N} : M\left(\frac{|a_i - a_j|}{\rho}\right) < \epsilon\right\}$.

Then $B^c \in I$.

We choose $k_0 \in B^c$.

Then for each $i, j \geq k_0$,

we have

$$\left\{k \in \mathbb{N} : M\left(\frac{|a_i - a_j|}{\rho}\right) < \epsilon\right\} \supseteq \left[\left\{k \in \mathbb{N} : M\left(\frac{|T(x_k^{(i)}) - T(x_k^{(j)})|}{\rho}\right) < \frac{\epsilon}{3}\right\}\right.$$
$$\cap \left\{k \in \mathbb{N} : M\left(\frac{| T(x_k^{(i)}) - a_i |}{\rho}\right) < \frac{\epsilon}{3}\right\}$$
$$\left.\cap \left\{k \in \mathbb{N} : M\left(\frac{| a_j - T(x_k^{(j)}) |}{\rho}\right) < \frac{\epsilon}{3}\right\}\right]$$

implies that

(a_i) is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar a in $\mathbb{C}$ such that $a_i \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then we show that, if
$U = \{k \in \mathbb{N} : M\left(\frac{|T(x_k^{(i)}) - a|}{\rho}\right) < \delta\}$, then $U^c \in I$.

Since $x^{(i)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \left\{ k \in \mathbb{N} : M\left(\frac{| T(x_k^{(q_0)}) - T(x_k) |}{\rho} \right) < \frac{\delta}{3} \right\} \tag{5.2.12}$$

implies $P^c \in I$.

The number q_0 can be chosen that together with (2.12), we have

$$Q = \left\{ k \in \mathbb{N} : M\left(\frac{| a_{q_0} - a |}{\rho} \right) < \frac{\delta}{3} \right\}$$

such that $Q^c \in I$.

Since $(x_k^{(i)}) \in \mathcal{M}_S^I(M)$, we have

$$\left\{ k \in \mathbb{N} : M\left(\frac{| T(x_k^{(q_0)}) - a_{q_0} |}{\rho} \right) \geq \delta \right\} \in I.$$

Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{ k \in \mathbb{N} : M\left(\frac{| T(x_k^{(q_0)}) - a_{q_0} |}{\rho} \right) < \frac{\delta}{3} \right\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where

$$U = \left\{ k \in \mathbb{N} : M\left(\frac{| T(x_k) - a |}{\rho} \right) < \delta \right\}$$

Therefore, for each $k \in U^c$, we have

$$\begin{aligned} \left\{ k \in \mathbb{N} : M\left(\tfrac{|T(x_k) - a|}{\rho} \right) < \delta \right\} \supseteq & \Bigg[\left\{ k \in \mathbb{N} : M\left(\tfrac{|T(x_k^{(q_0)}) - T(x_k)|}{\rho} \right) < \tfrac{\delta}{3} \right\} \\ & \cap \left\{ k \in \mathbb{N} : M\left(\frac{| a_{q_0} - a |}{\rho} \right) < \frac{\delta}{3} \right\} \\ & \cap \left\{ k \in \mathbb{N} : M\left(\frac{| T(x_k^{(q_0)}) - a_{q_0} |}{\rho} \right) < \frac{\delta}{3} \right\} \Bigg]. \end{aligned}$$

Then the result follows.

Since the inclusions $\mathcal{M}_S^I(M) \subset \mathcal{S}_\infty(M)$ and $\mathcal{M}_{S_0}^I(M) \subset \mathcal{S}_\infty(M)$ are strict so in view of Theorem (2.9) we have the following result.

Theorem 5.2.10. The spaces $\mathcal{M}_S^I(M)$ and $\mathcal{M}_{S_0}^I(M)$ are nowhere dense subsets of $\mathcal{S}_\infty(M)$.

CHAPTER 6

INTUITIONISTIC FUZZY ZWEIER I-CONVERGENT SEQUENCE SPACES

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CHAPTER-6

INTUITIONISTIC FUZZY ZWEIER I-CONVERGENT SEQUENCE SPACES

6.1. INTRODUCTION

The fuzzy theory has emerged as the most active area of research in many branches of science and engineering. Among various developments of the theory of fuzzy sets[109] (Zadeh et al., 1965) a progressive development has been made to find the fuzzy analogues of the classical set theory. In fact the fuzzy theory has become an area of active research for the last 50 years. It has a wide range of applications in the field of science and engineering, e.g. application of fuzzy topology in quantum particle physics that arises in string and $e^{(\infty)}$-theory of Naschie, El(Naschie et al.,[81]-[83]), chaos control, computer programming, nonlinear dynamical system and population dynamics etc.

In many branches of science and engineering we often come across with different type of sequences and certainly there are situations of inexactness where the idea of ordinary convergence does not work. So to deal with such situations we have to introduce new measures and tools which is suitable to the said situation.That is we are interested to put forward our studies in fuzzy like situations.

Here we recall some notations and basic definitions which we need throughout the work.

Definition 6.1. A binary operation $*$: $[0,1]\text{x}[0,1]\rightarrow[0,1]$ is said to be continuous t-norm if it satisfies the following conditions;

(a) $*$ is associative and commutative.

(b) $*$ is continuous.

(c)$a*1=a$ for all $a\in[0,1]$

(d)$a*c\leq b*d$ whenever $a\leq b$ and $c\leq d$ for each $a,b,c,d\in[0,1]$.

For example, $a*b=a.b$ is a continuous t-norm.

Definition 6.2. A binary operation $\diamond$; $[0,1]\text{x}[0,1]\rightarrow[0,1]$ is said to be continuous t-conorm if it satisfies the following conditions;

(a)$\diamond$ is associative and commutative.

(b) $\diamond$ is continuous.

(c)a$\diamond$0=a for all a$\in$[0,1]

(d)a$\diamond$c$\leq$b$\diamond$d whenever a$\leq$b and c$\leq$d for each a,b,c,d$\in$[0,1].

For example, a$\diamond$b=min$\{a+b,1\}$ is a continuous t-conorm.

Definition 6.3. The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions for every $x,y\in X$ and $s,t>0$:

(a) $\mu(x,t)+\nu(x,t)\leq 1$,

(b) $\mu(x,t)>0$,

(c) $\mu(x,t)=1$ if and only if $x=0$,

(d) $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,

(e) $\mu(x,t)*\mu(y,s)\leq\mu(x+y,t+s)$,

(f) $\mu(x,.):(0,\infty)\to[0,1]$ is continuous,

(g) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,

(h) $\nu(x,t)<1$,

(i) $\nu(x,t)=0$ if and only if $x=0$,

(j) $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,

(k) $\nu(x,t)\diamond\nu(y,s)\geq\nu(x+y,t+s)$,

(l) $\nu(x,.):(0,\infty)\to[0,1]$ is continuous,

(m) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm.

Definition 6.4. Let (X,μ,ν,*,$\diamond$) be IFNS and (x_n) be a sequence in X. Sequence (x_n) is said to be convergent to L *in* X with respecct to the intuitionistic fuzzy norm (μ,ν) if for every $\epsilon>0$ and $t>0$, there exists a positive integer n_0 such that $\mu(x_n$-L,t)> $1-\epsilon$ and $\nu(x_n$-L,t)< ϵ whenever $n>n_0$. In this case we write (μ,ν)-limx_n=L as $n\to\infty$.

Definition 6.5. If X be a non- empty set, then a family of set $I\subset P(X)(P(X)$

denoting the power set of X) is called an ideal in X if and only if
(a) $\phi \in I$;
(b) For each $A, B \in I$, we have $A \cup B \in I$;
(c) For each $A \in I$ and $B \subset A$ we have $B \in I$.

Definition 6.6. If X be a non- empty set. A non- empty family of sets $F \subset P(X)(P(X)$ denoting the power set of X) is called a filter on X if and only if
(a) $\phi \notin F$;
(b) For each $A, B \in F$, we have $A \cap B \in F$;
(c) For each $A \in F$ and $A \subset B$ we have $B \in F$.

Definition 6.7. Let $I \subset P(N)$ be a non trivial ideal and (X,μ,ν,*,$\diamond$) be an IFNS. A sequence $x = (x_n)$ of elements in X is said to be I- convergent to L *in* X with respect to the intuitionistic fuzzy norm (μ,ν) if for every $\epsilon > 0$ and $t > 0$, The set

$$\{n \in N : \mu(x_n - L, t) \geq 1 - \epsilon \text{ or } \nu(x_n - L, t) \leq \epsilon\} \in I$$

In this case L is called the I-limit of the sequence (x_n) with respecct to the intuitionistic fuzzy norm (μ,ν) and we write $I_{(\mu,\nu)}$- limx_n=L.

Definition 6.8. Let (X,μ,ν,*,$\diamond$) be an IFNS. Let $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_x(r,t) = \{y \in X : \{k \in N : \mu(x_k - L, t) \leq 1 - r \text{ or } \nu(x_k - L, t) \geq r\} \in I\}$$

is called an open ball with center x and radius r with respect to t.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar [5], Malkowsky et al.[74], Ng and Lee et al.[85], and Wang et al.[106],Şengönület al.[99] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1 - p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1$, $1 < p < \infty$ and $\mathcal{Z}^p$ denotes the matrix $\mathcal{Z}^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Analogous to Başar and Altay et al. [5] (2003), Şengönül et al.[99] (2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c\};$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c_0\}.$$

Recently Khan, Ebadullah and Yasmeen et al. [55](2014) introduced the following classes of sequences

$\mathcal{Z}^I = \{(x_k) \in \omega :$ there exists $L \in \mathbb{C}$ such that for a given $\varepsilon > 0, \{k \in \mathbb{N} : |x_k^{/} - L| \geq \varepsilon\} \in I\}$;

$\mathcal{Z}_0^I = \{(x_k) \in \omega :$ for a given $\varepsilon > 0, \{k \in \mathbb{N} : |x_k^{/}| \geq \varepsilon\} \in I\}$,

where $(x_k^{/}) = (\mathcal{Z}^p x)$

In this chapter we introduce the intuitionstic fuzzy Zweier I-convergent sequence spaces as follows.

$$\mathcal{Z}^I_{(\mu,\nu)} = \{(x_k) \in \omega : \{k \in N : \mu(x_k^{/} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_k^{/} - L, t) \geq \epsilon\} \in I\},$$
$$\mathcal{Z}^I_{0(\mu,\nu)} = \{(x_k) \in \omega : \{k \in N : \mu(x_k^{/}, t) \leq 1 - \epsilon \text{ or } \nu(x_k^{/}, t) \geq \epsilon\} \in I\}.$$

6.2. MAIN RESULTS

Theorem.6.2.1. $\mathcal{Z}^I_{(\mu,\nu)}$ and $\mathcal{Z}^I_{0(\mu,\nu)}$ are linear spaces.

Proof. We prove the result for $\mathcal{Z}^I_{(\mu,\nu)}$. Similarly the result can be proved for $\mathcal{Z}^I_{0(\mu,\nu)}$.
Let $(x_k^{/}), (y_k^{/}) \in \mathcal{Z}^I_{(\mu,\nu)}$ and let α, β be scalars. Then for a given $\epsilon > 0$.
For $\alpha = 0, \beta = 0$ the proof is trivial.
we have

$$A_1 = \{k \in N : \mu(x_k^{/} - L_1, \frac{t}{2|\alpha|}) \leq 1 - \epsilon \text{ or } \nu(x_k^{/} - L_1, \frac{t}{2|\alpha|}) \geq \epsilon\} \in I$$

$$A_2 = \{k \in N : \mu(y_k^{/} - L_2, \frac{t}{2|\beta|}) \leq 1 - \epsilon \text{ or } \nu(y_k^{/} - L_2, \frac{t}{2|\beta|}) \geq \epsilon\} \in I$$

$$A_1^c = \{k \in N : \mu(x_k^{/} - L_1, \frac{t}{2|\alpha|}) > 1 - \epsilon \text{ or } \nu(x_k^{/} - L_1, \frac{t}{2|\alpha|}) < \epsilon\} \in F(I)$$

$$A_2^c = \{k \in N : \mu(y_k^{/} - L_2, \frac{t}{2|\beta|}) > 1 - \epsilon \text{ or } \nu(y_k^{/} - L_2, \frac{t}{2|\beta|}) < \epsilon\} \in F(I)$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I$. It follows that A_3^c is a non-empty set in F(I).

We shall show that for each $(x_k^{/}), (y_k^{/}) \in \mathcal{Z}^I_{(\mu,\nu)}$

$$A_3^c \subset \{k \in N : \mu((\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2), t) > 1 - \epsilon \text{ or } \nu((\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2), t) <$$

Let $m \in A_3^c$. In this case

$$\mu(x_m^{/} - L_1, \frac{t}{2|\alpha|}) > 1 - \epsilon \text{ or } \nu(x_m^{/} - L_1, \frac{t}{2|\alpha|}) < \epsilon$$

and

$$\mu(y_m^{/} - L_2, \frac{t}{2|\beta|}) > 1 - \epsilon \text{ or } \nu(y_m^{/} - L_2, \frac{t}{2|\beta|}) < \epsilon$$

We have

$$\begin{aligned}
\mu((\alpha x_m^{/} + \beta y_m^{/}) - (\alpha L_1 + \beta L_2), t) &\geq \mu(\alpha x_m^{/} - \alpha L_1, \frac{t}{2}) * \mu(\beta y_m^{/} - \beta L_2, \frac{t}{2}) \\
&= \mu(x_m^{/} - L_1, \frac{t}{2|\alpha|}) * \mu(y_m^{/} - L_2, \frac{t}{2|\beta|}) \\
&> (1-\epsilon) * (1-\epsilon) \\
&= (1-\epsilon)
\end{aligned}$$

and

$$\begin{aligned}
\nu((\alpha x_m^{/} + \beta y_m^{/}) - (\alpha L_1 + \beta L_2), t) &\leq \nu(\alpha x_m^{/} - \alpha L_1, \frac{t}{2}) \diamond \nu(\beta y_m^{/} - \beta L_2, \frac{t}{2}) \\
&= \nu(x_m^{/} - L_1, \frac{t}{2|\alpha|}) \diamond \nu(y_m^{/} - L_2, \frac{t}{2|\beta|}) \\
&< \epsilon \diamond \epsilon \\
&= \epsilon
\end{aligned}$$

This implies that

$$A_3^c \subset \{k \in N : \mu((\alpha x_k^{/}+\beta y_k^{/})-(\alpha L_1+\beta L_2),t) > 1-\epsilon \text{ or } \nu((\alpha x_k^{/}+\beta y_k^{/})-(\alpha L_1+\beta L_2),t) < \epsilon\}$$

Hence $\mathcal{Z}^I_{(\mu,\nu)}$ is a linear space.

Theorem 6.2.2. Every open ball $B_{x^/}(r,t)$ is an open set in $\mathcal{Z}^I_{(\mu,\nu)}$.

Proof. Let $B_{x^/}(r,t)$ be an open ball with centre $x^/$ and radius r with respect to t. That is

$$B_{x^/}(r,t) = \{y \in X : \{k \in N : \mu(x_{k^/} - L,t) \leq 1-r \text{ or } \nu(x_{k^/} - L,t) \geq r\} \in I\}.$$

Let $y^/ \in B^c_{x^/}(r,t)$. Then $\mu(x^/ - y^/,t) > 1-r$ and $\nu(x^/ - y^/,t) < r$.
Since $\mu(x^/ - y^/,t) > 1-r$, there exists $t_0 \in (0,t)$ such that $\mu(x^/ - y^/,t_0) > 1-r$ and $\nu(x^/ - y^/,t_0) < r$.
Putting $r_0 = \mu(x^/ - y^/,t_0)$. we have $r_0 > 1-r$, there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$.
For $r_0 > 1-s$ we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1-s$ and $(1-r_0)\diamond(1-r_2) \leq s$.
Putting $r_3 = max\{r_1, r_2\}$, consider the ball $B^c_{y^/}(1-r_3, t-t_0)$
We prove that $B^c_{y^/}(1-r_3,t-t_0) \subset B^c_{x^/}(r,t)$.
Let $z^/ \in B^c_{y^/}(1-r_3,t-t_0)$.
$\mu(y^/ - z^/, t-t_0) > r_3$ and $\nu(y^/ - z^/, t-t_0) < r_3$.
Therefore

$$\mu(x^/-z^/,t) \geq \mu(x^/-y^/,t_0)*\mu(y^/-z^/,t-t_0) \geq (r_0*r_3) \geq (r_0*r_1) \geq (1-s) > (1-r)$$

and

$$\nu(x^/-z^/,t) \leq \nu(x^/-y^/,t_0)\diamond\nu(y^/-z^/,t-t_0) \leq (1-r_0)\diamond(1-r_3) \leq (1-r_0)\diamond(1-r_2) \leq s < r.$$

Thus $z^/ \in B^c_{x^/}(r,t)$ and hence $B^c_{y^/}(1-r_3,t-t_0) \subset B^c_{x^/}(r,t)$.

Remark $\mathcal{Z}^I_{(\mu,\nu)}$ is an IFNS.
Define

$$\tau_{(\mu,\nu)} = \{A \subset \mathcal{Z}^I_{(\mu,\nu)} : \text{for each } x \in A \text{ there exists } t > 0 \text{ and } r \in (0,1) \text{ such that } B_{x^/}(y^/,t) \subset A\}$$

Then $\tau_{(\mu,\nu)}$ is a topology on $\mathcal{Z}^I_{(\mu,\nu)}$.

Theorem 6.2.3. The topology $\tau_{(\mu,\nu)}$ on $\mathcal{Z}^I_{0(\mu,\nu)}$ is first contable.

Proof. $\{B_{x'}(\frac{1}{n},\frac{1}{n}) : \text{n=1, 2, 3 ....}\}$ is a local base at x', the topology $\tau_{(\mu,\nu)}$ on $\mathcal{Z}^I_{0(\mu,\nu)}$ is first contable

Theorem 6.2.4. $\mathcal{Z}^I_{(\mu,\nu)}$ and $\mathcal{Z}^I_{0(\mu,\nu)}$ are Hausdorff spaces.

Proof. We prove the result for $\mathcal{Z}^I_{(\mu,\nu)}$. Smilarly the proof follows for $\mathcal{Z}^I_{0(\mu,\nu)}$.

Let $x', y' \in \mathcal{Z}^I_{(\mu,\nu)}$ such that $x' \neq y'$.
Then $0 < \mu(x'-y',t) < 1$ and $0 < \nu(x'-y',t) < 1$.
Putting $r_1 = \mu(x'-y',t)$ and $r_2 = \nu(x'-y',t)$ and $r = max\{r_1, 1-r_2\}$.
For each $r_0 \in (r,1)$, there exists r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and
$(1-r_3) \diamond (1-r_4) \leq (1-r_0)$.
Putting $r_5 = max\{r_3, 1-r_4\}$ and consider the open balls
$B_{x'}(1-r_5,\frac{t}{2})$ and $B_{y'}(1-r_5,\frac{t}{2})$.
Then clearly $B^c_{x'}(1-r_5,\frac{t}{2}) \cap B^c_{y'}(1-r_5,\frac{t}{2}) = \phi$
For if there exists $z' \in B^c_{x'}(1-r_5,\frac{t}{2}) \cap B^c_{y'}(1-r_5,\frac{t}{2})$ then

$$r_1 = \mu(x'-y',t) \geq \mu(x'-z',\frac{t}{2}) * \mu(z'-y',\frac{t}{2}) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1$$

and

$$r_2 = \nu(x'-y',t) \leq \nu(x'-z',\frac{t}{2}) \diamond \nu(z'-y',\frac{t}{2}) \leq (1-r_5)\diamond(1-r_5) \leq (1-r_4)\diamond(1-r_4) \leq (1-r_0)$$

which is a contradiction, hence $\mathcal{Z}^I_{(\mu,\nu)}$ is Hausdorff.

Theorem 6.2.5. $\mathcal{Z}^I_{(\mu,\nu)}$ is an IFNS. $\tau_{(\mu,\nu)}$ is a topology on $\mathcal{Z}^I_{(\mu,\nu)}$.Then a sequence $(x'_k) \in \mathcal{Z}^I_{(\mu,\nu)}$, $x'_k \to x'$ if and only if $\mu(x'_k - x',t) \to 1$ and $\nu(x'_k - x',t) \to 0$ as $k \to \infty$.

Proof. Fix $t_0 > 0$. Suppose $x'_k \to x'$. Then for $r \in (0,1)$, there exists $n_0 \in N$ such that $x'_k \in B_{x'}(r,t)$ for all $k \geq n_0$.

$$B_{x'}(r,t) = \{k \in N : \mu(x'_k - x',t) \leq 1-r \text{ or } \nu(x'_k - x',t) \geq r\} \in I$$

Such that $B^c_{x'}(r,t) \in F(I)$
Then $1-\mu(x'_k - x', t) < r$ and $\nu(x'_k - x', t) < r$.
Hence $\mu(x'_k - x', t) \to 1$ and $\nu(x'_k - x', t) \to 0$ as $k \to \infty$.

Conversely, if for each $t > 0$ $\mu(x'_k - x', t) \to 1$ and $\nu(x'_k - x', t) \to 0$ as $k \to \infty$, then for $r \in (0,1)$, there exists $n_0 \in N$ such that $1-\mu(x'_k - x', t) < r$ and $\nu(x'_k - x', t) < r$. for all $k \geq n_0$. it follows that
$\mu(x'_k - x', t) > 1 - r$ and $\nu(x'_k - x', t) < r$ for all $k \geq n_0$.
Thus $x'_k \in B^c_{x'}(r,t)$ for all $k \geq n_0$ and hence $x'_k \to x'$.

Theorem 6.2.6. A sequence $x = (x'_k) \in \mathcal{Z}^I_{(\mu,\nu)}$ I-converges if and only if for every $\epsilon > 0$ and $t > 0$ there exists a number $N = N(x, \epsilon, t)$ such that

$$\{k \in N : \mu(x'_k - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(x'_k - L, \frac{t}{2}) < \epsilon\} \in F(I)$$

Proof. Suppose that $I_{(\mu,\nu)} - x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1-\epsilon)*(1-\epsilon) > 1-s$ and $\epsilon \diamond \epsilon < s$.Then for each $x \in \mathcal{Z}^I_{(\mu,\nu)}$,

$$A_x(\epsilon, t) = \{k \in N : \mu(x'_k - L, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(x'_k - L, \frac{t}{2}) \geq \epsilon\} \in I$$

which implies that

$$A^c_x(\epsilon, t) = \{k \in N : \mu(x'_k - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(x'_k - L, \frac{t}{2}) < \epsilon\} \in F(I).$$

Conversly let us choose $N \in A^c_x(\epsilon, t)$. Then

$$\mu(x'_N - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(x'_N - L, \frac{t}{2}) < \epsilon$$

Now we want to show that there exists a number $N = N(x, \epsilon, t)$ such that

$$\{k \in N : \mu(x'_k - x'_N, t) \leq 1 - s \text{ or } \nu(x'_k - x'_N, t) \geq s\} \in I$$

For this, define for each $x \in \mathcal{Z}^I_{(\mu,\nu)}$

$$B_x(\epsilon, t) = \{k \in N : \mu(x'_k - x'_N, t) \leq 1 - s \text{ or } \nu(x'_k - x'_N, t) \geq s\} \in I$$

Now we show that $B_x(\epsilon, t) \subset A_x(\epsilon, t)$.

Suppose that $B_x(\epsilon,t) \not\subset A_x(\epsilon,t)$.

Then there exists $n \in B_x(\epsilon,t)$ and $n \notin A_x(\epsilon,t)$. Therefore we have

$$\mu(x_n^/ - x_N^/, t) \leq 1-s \text{ and } \mu(x_n^/ - L, \frac{t}{2}) > 1-\epsilon$$

In particular $\mu(x_N^/ - L, \frac{t}{2}) > 1-\epsilon$.
Therefore we have

$$1-s \geq \mu(x_n^/ - x_N^/, t) \geq \mu(x_n^/ - L, \frac{t}{2}) * \mu(x_N^/ - L, \frac{t}{2}) \geq (1-\epsilon) * (1-\epsilon) > 1-s$$

which is not possible. On the other hand

$$\nu(x_n^/ - x_N^/, t) \geq s \text{ and } \nu(x_n^/ - L, \frac{t}{2}) < \epsilon$$

In particular $\nu(x_N^/ - L, \frac{t}{2}) < \epsilon$.
Therefore we have

$$s \leq \nu(x_n^/ - x_N^/, t) \leq \nu(x_n^/ - L, \frac{t}{2}) \diamond \nu(x_N^/ - L, \frac{t}{2}) \leq \epsilon \diamond \epsilon < s$$

which is not possible.
Hence $B_x(\epsilon,t) \subset A_x(\epsilon,t)$.

$A_x(\epsilon,t) \in I$ implies $B_x(\epsilon,t) \in I$.
Hence the proof.

LIST OF PUBLICATIONS

[1] *On I-convergent sequence spaces defined by compact operator and modulus function*
Cogent Mathematics,(UK)
Vol. 2015 (13 pages)

[2] *On BV_σ I-convergent sequence spaces defined by an Orlicz function.*
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[3] *On Some I-Convergent Sequence Spaces Defined by a Compact operator and an Orlicz Function.*
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[4] Intuitionistic fuzzy Zweier I-convergent sequence spaces.
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[5] *On Paranorm I-Convergent Seequence Spaces of Bounded Operators Defined by a Modulus function.*
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